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Semi-Precontinuous Multi Functions

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ABSTRACT

The concept of single-valued semi-precontinuous function has been defined by Przemski [7]. In this paper single-valued semi-precontinuous function has been extended to multifunction. Several characterisations and basic properties of this multifunction have been obtained here.

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1. Introduction

In 1932, Kuratowski [4] and Bouligand [3] independently introduced two kinds of semi-continuity of a multifunction which are termed as upper semi-continuous and lower semi-continuous. R. E. Smithson in [8] has published a survey of some results on multifunctions. He has elegantly pointed out the difference between the theory of single-valued functions and that of multifunctions. R.E.Smithson [8], V. Popa [6], P.Martiz [5] and many others have investigated multifunctions extensively. In 1986, Andrijević [1] introduced the notion of semi-preopen sets and in 1992, Przemski [7] defined semi-precontinuity as single-valued function utilising semi-preopen sets. In this paper the single-valued semi-precontinuous functions is extended to semi-precontinuous multifunction.

2. Preliminaries

Throughout the paper (X, τ) or simply X always denotes nontrivial topological spaces. The closure (resp. interior) of the subset A is denoted by $Cl_X(A)$ (resp. $Int_X(A)$) .

Definition 2.1. [1] In (X, τ), $A \subset X$ is called

(i) a semi-preopen set (briefly s.p.o. set) iff $A \subset Cl(Int(Cl(A)))$.

The family of all s.p.o. sets is denoted by SPO(X). For each $x \in X$, the family of all s.p.o. sets containing x is denoted by SPO(X, x).

Definition 2.2.[1] The complement of a s.p.o. set is called semi-preclosed. Equivalently a set F is semi-preclosed iff $Int(Cl(Int(A))) \subset F$. The family of all semi-preclosed sets is denoted by SPF (X).

Definition 2.3. [7] A single valued function $f : X \to Y$ is said to be semi- precontinuous if the inverse image of every open set in Y is semi-preopen in X

Definition 2.4 .[1] The semi-preclosure (resp,semi-preinterior) of $A \subset X$ is denoted by spcl (A)(resp.spint(A)) and is defined by $spcl(A) = \cap\{B : B$ is semi-preclosed and $B \supset A\}$(resp.$spint(A) = \cup \{B: B$ is semi-preopen and $B \subset A\}$.

A multifunction from a set X to a set Y is a function from X to $2^Y - \{\phi\}$ where 2^Y is the power set of Y. For a multifunction $F : X \to Y$, $F^+[B]$ and $F^-[B]$, respectively denote the upper and lower inverses of the set $B \subset Y$, where $F^+[B] = \{x \in X : F(x) \subset B\}$ and $F^-[B] = \{x \in X : F(x) \cap B \neq \phi\}$.In this paper, single-valued functions are denoted by lower case letters while upper case letters are used to denote a multifunction.

The following elementary results have been frequently utilised in this paper.

Lemma 2.1. [2] For any multifunction $F : X \to Y$

1. $F^+[Y - B] = X - F^-[B]$, $B \subset Y$;
2. $F^-[Y - B] = X - F^+[B]$, $B \subset Y$;
3. $A \subset F^+[F[A]]$, $A \subset X$;
4. $F[F^+[B]] \subset B$, $B \subset Y$;
5. $B \subset C \subset Y \Rightarrow F^+[B] \subset F^+[C]$, $F^-[B] \subset F^-[C]$;
6. $A \subset C \subset X \Rightarrow F[A] \subset F[C]$;
7. $F[A] \cap B \subset F[A \cap F^-[B]]$, $A \subset X$, $B \subset Y$;
8. If $A_\alpha \subset X \ \forall \alpha \in \Lambda$, then $F[\bigcup_{\alpha \in \Lambda} A_\alpha] = \bigcup_{\alpha \in \Lambda} F[A_\alpha]$.

Definition 2.5. [8] A multifunction $F : X \to Y$ is sad to be

1. upper semi-continuous (briefly usc) iff for each closed set $A \subset Y$, $F^-[A]$ is a closed subset of X,
2. lower semi-continuous (briefly lsc) iff for each open set $U \subset Y$, $F^-[U]$ is an open subset of X,
3. continuous iff F is both lsc and usc.

3. Semi-Precontinuous Multifunction

Upper (lower) semi-precontinuous multifunction is introduced as follows:

Definition 3.1. A multifunction $F : X \to Y$ is termed upper semi-precontinuous (resp. lower semi-precontinuous), briefly uspc (resp. lspc), iff for each closed (resp. open) set $A \subset Y$, $F^-[A] \in SPF(X)$ (resp. $F^-[A] \in SPO(X)$).

Definition 3.2. A multifunction $F : X \to Y$ is semi-precontinuous (briefly spc) iff F is both uspc and lspc.

Remark 3.1. Clearly Definition 3.1 is the equivalent form of the following definition

Definition 3.3. A multifunction $F : X \to Y$ is said to be

1. uspc at $x_0 \in X$ iff for each $V \in \Sigma(F(x_0))$ in Y there is a $U \in SPO(X, x_0)$ with $F(x) \subset V \quad \forall x \in U$;
2. uspc iff F is uspc at each point of X;
3. lspc at $x_0 \in X$ iff every open set $V \subset Y$ with $F(x_0) \cap V \neq \phi$, there exists a $U \in SPO(X, x_0)$ such that $F(x) \cap V \neq \phi \quad \forall x \in U$;
4. lspc iff F is lspc at each point of X;
5. spc iff F is both lspc and uspc.

Remark 3.2. For a single-valued function the concepts uspc and lspc coincide. But this is not the case for spc multifunction as is seen from the following example.

Example 3.1. Let X = Y = [0, 1] be equipped with the usual topology. Let $F : X \to Y$ be defined by

$$F(x) = \{x/2\} \quad \text{for } 0 \le x < 1/2,$$
$$= [1/4, 3/4] \quad \text{for } x = 1/2,$$
$$= \{(x+1)/2\} \text{ for } 1/2 < x \le 1.$$

Closed-ness and hence semi-preclosedness of $F^-[A]$ in X can easily be checked for every closed set A of Y .So F is uspc. But $F^-[(3/8, 5/8)] = \{1/2\} \notin SPO(X)$ for the open set $(3/8, 5/8)$ of Y and so F is not lspc.

Remark 3.3. Since openness implies semi-preopenness every usc (resp. lsc) multifunction is uspc (resp. lspc). But the converse is not, in general true. This can be checked from the following example.

Example 3.2. Let X and Y be the same sets of Example 3.1, τ_X= the discrete topology on X while τ_Y = the point inclusion topology with the including point 0. Let

$F : (X, \tau_X) \to (Y, \tau_Y)$ be defined by $F(x) = Y \ \forall x \in X - \{1\}$, $F(1) = \{1\}$.

The indiscreteness of τ_X produces SPO (X) = P (X) the power set of X. As a result $F^-[A] \in SPF(X)$ for all $A \in F(\tau_Y)$ and $F^-[A] \in SPO(X)$ for all $A \in \tau_Y$. This gives that F is both uspc and lspc. Hence F is a spc- multifunction. Let A = {1/2}, B = {0} in Y, then $A \in F(\tau_Y)$ and $B \in \tau_Y$. But $F^-[A] = [0, 1) = F^-[B]$ which is neither open nor closed. Consequently, F is neither usc nor lsc. Hence F is not a continuous multifunction.

The following theorem offers a set of characterisations of uspc multifunction.

Theorem 3.1. For a multifunction $F : X \to Y$ the following statements are equivalent :

a) F is uspc;
b) $F^+[A] \in SPO(X)$ for every open $A \subset Y$;
c) $F^+[Int_Y(B)] \subset Cl_X(Int_X(Cl_X(F^+[B])))$ for every $B \subset Y$;
d) $F^+[Int_Y(C)] \subset spint_X(F^+[C])$ for every $C \subset Y$;
e) $Int_X(Cl_X(Int_X(F^-[D]))) \subset F^-[Cl_Y(D)]$ for every $D \subset Y$;
f) $spcl_X(F^-[E]) \subset F^-[Cl_Y(E)]$, for every $E \subset Y$.

Proof. (a) ⇒ (b). Let F be uspc. Suppose A be open in Y. Then Y – A is a closed subset of Y. The uspc of F, then induces that $F^-[Y - A] \in SPF(X)$.

By Lemma2.1, $X - F^+[A] = F^-[Y - A] \in SPF(X) \Rightarrow F^+[A] \in SPO(X)$.

(b) ⇒ (c). Let $B \subset Y$ and $A = Int_Y(B)$. Then A is open in Y. By (b),$F^+[A] \in SPO(X)$. $\Rightarrow F^+[Int_Y(B)] \in SPO(X) \Rightarrow F^+[Int_Y(B)] \subset Cl_X(Int_X(Cl_X(F^+[nt_Y(B)])$ …(1)

Again, the inclusion $Int_Y(B) \subset B$ gives, by Lemma2.1, $F^+[Int_Y(B)] \subset F^+[B]$.

Hence (1) reduces to $F^+[Int_Y(B)] \subset Cl_X(Int_X(Cl_X(F^+[B])))$.

(c) ⇒ (d). For $C \subset Y$, the inclusion $F^+[Int_Y(C)] \subset F^+[C]$ holds in view of Lemma2.1.

Also by (c) $F^+[Int_Y(C)] \subset Cl_X(Int_X(Cl_X(F^+[C])))$. Hence $F^+[Int_Y(C)] \subset F^+[C] \cap Cl_X(Int_X(Cl_X(F^+[C]))) \Rightarrow F^+[Int_Y(C)] \subset spint(F^+[C])$.

(d) ⇒ (b). Let $A \subset Y$ be open. By (d),$F^+[A] = F^+[Int_Y(A)] \subset spint_X(F^+[A])$

$\Rightarrow F^+[A] = spint_X(F^+[A])$. Thus $F^+[A] \in SPO(X)$.

(b) ⇒ (a). Let $V \subset Y$ be closed and A = Y – V.Thus A is open in Y. So, by (b)$F^+[Y - V] = F^+[A] \in SPO(X)$.This, by Lemma2.1, produces that $X - F^-[V] \in SPO(X) \Rightarrow F^-[V] \in SPO(X) \Rightarrow$ F is uspc.

(a) ⇒ (c). Suppose $D \subset Y$. Then $Cl_Y(D)$ is a closed set in Y. By (a),

$F^-[Cl_Y(D)] \in SPF(X) \Rightarrow Int_X(Cl_X(Int_X(F^-[Cl_Y(D)]))) \subset F^-[Cl_Y(D)]$ …(2)

By Lemma2.1, $F^-[D] \subset F^-[Cl_Y(D)] \Rightarrow Int_X(Cl_X(Int_X(F^-[D])))$

$\subset Int_X(Cl_X(Int_X(F^-[Cl_Y(D)])))$ which $Int_X(Cl_X(Int_X(F^-[D]))) \subset F^-[Cl_Y(D)]$.

(e) ⇒ (f). Suppose $E \subset Y$. Then by (c),$Int_X(Cl_X(Int_X(F^-[D]))) \subset F^-[Cl_Y(D)]$

$\Rightarrow F^-[D] \cup Int_X(Cl_X(Int_X(F^-[D]))) \subset F^-[Cl_Y(D)] \Rightarrow spcl(F^-[D]) \subset F^-[Cl_Y(D)]$.

(f) ⇒ (a). Suppose $V \subset Y$ be closed. By (f),$spcl_X(F^-[V]) \subset F^-[Cl_Y(V)] = F^-[V]$

$\Rightarrow spcl_X(F^-[V]) = F^-[V]$.Thus $F^-[V] \in SPF(X)$ .So, F is uspc.

Theorem 3.2. For a multifunction $F : X \to Y$ the following are equivalent

(a) F is lspc;
(b) $F^+[A] \in SPF(X)$ for every closed set $A \subset Y$;
(c) $F^-[Int_Y(B)] \subset Cl_X(Int_X(Cl_X(F^-[B]))$ for every $B \subset Y$;
(d) $F^-[Int_Y(C)] \subset spint_X(F^-[C])$ for every $C \subset Y$;
(e) $Int_X(Cl_X(Int_X(F^+[D]))) \subset F^+[Cl_Y(D)]$ for every $D \subset Y$;
(f) $spcl_X(F^+[E]) \subset F^+[Cl_Y(E)]$ for every $E \subset Y$;
(g) $F[spcl_X(G)] \subset Cl_Y(F[G])$ for every $G \subset X$;
(h) $F[Int_X(Cl_X(Int_X(H)))] \subset Cl_Y(F[H])$ for every $H \subset X$.

Proof. (a) ⇒ (b). $A \subset Y$ be closed. Then Y – A is open in Y. The lspc of F gives that

$F^-[Y - A] \in SPO(X)$.Thus , $X - F^+[A] = F^-[Y - A] \in SPO(X) \Rightarrow F^+[A] \in SPF(X)$.

(b) ⇒ (c). Clearly $Y - Int_Y(B)$ is closed in Y. By (b), $F^+[Y - Int_Y(B)] \in SPF(X)$.

This implies that $\Rightarrow X - F^-[Int_Y(B)] \in SPF(X) \Rightarrow F^-[Int_Y(B)] \in SPO(X) \Rightarrow F^-[Int_Y(B)] \subset Cl_X(Int_X(Cl_X(F^-[Int_Y(B)])))$. Again by Lemma2.1, $F^-[Int_Y(B)] \subset F^-[B]$.

Hence from the foregoing one obtains $F^-[Int_Y(B)] \subset Cl_X(Int_X(Cl_X(F^-[B])))$.

(c) ⇒ (d). Pursuing the same reasoning applied for (c) ⇒ (d) in the previous theorem, we obtain this implication.

(d) ⇒ (e). By (d), for any $D \subset Y$, $F^-[Int_Y(Y - D)] \subset spint_X F^-[Y - D]$

which ensures that $F^{-}[Int_Y(Y - D)] \subset Cl_X(Int_X(Cl_X(F^{-}[Y - D])))$. On taking complements we observe $X - F^{-}[Int_Y(Y - D)] \supset X - Cl_X(Int_X(Cl_X(F^{-}[Y - D])))$.

By Lemma2.1, $F^{+}[Y - Int_Y(Y - D)] \supset Int_X(X - Int_X(Cl_X(F^{-}[Y - D])))$

$\Rightarrow F^{+}[Cl_Y(D)] \supset Int_X(Cl_X(X - Cl_X(F^{-}[Y - D]))) = Int_X(Cl_X(Int_X(X - F^{-}[Y - D])))$

$= Int_X Cl_X(Int_X(F^{+}[D])))$. Thus $Int_X(Cl_X(Int_X(F^{+}[D]))) \subset F^{+}[Cl_Y(D)]$.

(e) $\Rightarrow$ (f). By (e), for any $E \subset Y$, $Int_X(Cl_X(Int_X(F^{+}[E]))) \subset F^{+}[Cl_Y(E)]$. So, $F^{+}[E] \subset F^{+}[Cl_Y(E)]$. Thus $F^{+}[E] \cup Int_X(Cl_X(Int_X(F^{+}([E]))) \subset F^{+}[Cl_Y(E)]$ hence $spcl_X(F^{+}[E]) \subset F^{+}[Cl_Y(E)]$.

(f) $\Rightarrow$ (g). Suppose $G \subset X$ and $V = F(G)$. By definition $V = F[G] = \cup \{F(x) : x \in G\}$.

Thus $x \in G \Rightarrow F(x) \subset V \Rightarrow x \in F^{+}[V]$ and so, $G \subset F^{+}[V]$. So, $spcl_X(G) \subset spcl_X(F^{+}[V])$.

By (f), then $spcl_X(G) \subset spcl_X(F^{+}[V]) \subset F^{+}[Cl_Y(V)]$. So by Lemma2.1 $F[spcl_X(G)] \subset F F^{+}[Cl_Y(V)] \subset Cl_Y(V) = Cl_Y(F[G])$.

(g) $\Rightarrow$ (h). Suppose $H \subset X$. By (g), $F[spcl_X(H)] \subset Cl_Y(F[H]$

$\Rightarrow F[H \cup Int_X(Cl_X(Int_X(H))) \subset Cl_Y(F[H]) \Rightarrow F[Int_X(Cl_X(Int_X(H)))] \subset Cl_Y(F[H])$

(h) $\Rightarrow$ (a). Suppose U be any open set. Then $Y - U$ is closed in Y. Let $H = F^{+}[Y - U]$. Since $F(x) \subset Y - U$, for all $x \in H$ it follows that $\cup \{F(x): x \in H\} \subset Y - U \Rightarrow F[H] \subset Y - U$. By (h), $F[Int_X(Cl_X(Int_X(H)))] \subset Cl_Y(F[H]) \subset Cl_Y(Y - U) \Rightarrow F[Int_X(Cl_X(Int_X(H)))] \subset Y - U$. Hence by Lemma2.1 the foregoing gives $F^{+} F[Int_X(Cl_X(Int_X(H)))] \subset F^{+}[Y - U] \Rightarrow Int_X(Cl_X(Int_X(F^{+}[Y - U]))) \subset F^{+}[Y - U] \Rightarrow F^{+}[Y - U] \in SPF(X)$

$\Rightarrow X - F^{-}[U] \in SPF(X)$ whence by Lemma2.1, $F^{-}[U] \in SPO(X)$. Hence F is lspc.

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