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A Review of Fixed Point Theorems with definition and proof

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ABSTRACT

This paper gives the fundamental suppositions which are utilized in the verifications of such theorems when they are worried about capacities that are contractive somehow or another. We present a general settled point theorem which can be viewed as the fundamental standard of evidence for settled point theorems. Additionally we contemplate two sorts of general coincidence guide theorems and their applications toward metric spaces and ultrametric spaces. Further, we locate an elective way to deal with coincidence point theorems. At last, we have contemplated the settled point theorems with definition and confirmation with the legitimate connection between these theorems.

1. Introduction

Coincidence point theorems concern two functions f, g from a set X into another set Y that, under certain conditions, admit a coincidence point. A coincidence point is a component x ∈ X to such an extent that its pictures under the capacities f, g are the equivalent; as it were, fx = gx. Settled point theorems think of one as capacity f from a set X into itself and give conditions for the presence of a settled point, that is, a component x ∈ X with the end goal that fx = x. Settled point theorems can be considered as uncommon instances of occurrence point theorems where X = Y and the second capacity g is the character. The presence of a settled purpose of a capacity characterized on some space X communicates some sort of culmination of this space. The established Banach Fixed Point Theorem requires the culmination of the measurement space on which the contractive capacity is characterized. A comparative outcome for ultrametric spaces was gotten by S. Prieß-Crampe (see [9]) and after that summed up by S. Prieß-Crampe and P. Ribenboim (see [10], and [11]).

For this situation the required fulfillment property is called round culmination. The traditional Brouwer Fixed Point Theorem requires minimization of a topological space, which is identical to a simple of round fulfillment, now connected to the accumulation of all nonempty shut subsets of this space. As of late Franz-Viktor Kuhlmann and Katarzyna Kuhlmann [4], [5], [6], and [7] have built up a general structure for settled point theorems which work with contractive capacities. This general system considers capacities characterized on ball spaces that are circularly total. The primary focal point of this postulation is to separate a general rule of evidence that works with settled point theorems and coincidence point theorems. In this proposal, I incorporate a portion of the work done by Franz-Viktor Kuhlmann and Katarzyna Kuhlmann [4], [5], [6], and [7] in Section 4, and apply the general settled direct theorems toward metric and ultrametric spaces.

The second and third segments are committed to definitions, apparatuses, and foundation material that are required in the later segments. We present a few ideas about arranged sets in Section 2. Additionally, we talk about measurement, ultrametric, and topological spaces. Segment 3 comprises of two segments. In the principal segment, we present ball spaces and their grouping, and we investigate the properties of capacities on ball spaces in the second segment. All through these two areas, we give instances of the ideas for more clarification. In the primary segment of Section 4, we present a general settled point theorem that works with capacities f that are characterized on roundly total ball spaces, and the balls B in the space are f-shut, i.e., f(B) ⊆ B. We give the evidence of this theorem and utilize it to locate a settled point for a capacity that is characterized on a ball space whose components are f-contracting (see Definition 3.2.2). After that we apply the general settled direct theorem toward metric spaces (Banach's Theorem), and ultametric spaces (Theorem of Prieß-Crampe and Ribenboim). In the second segment, we present a Bx-type settled point theorem. In this theorem, a capacity on a ball space is should have been self-contractive (see Definition 3.2.5) so as to have a settled point on the space. As in the main segment, we give the utilization of the Bx-type settled direct theorem toward metric and ultrametric spaces to demonstrate the theorems of Banach and of Prieß-Crampe and Ribenboim.

In the last segment, we acquaint an elective methodology with coincidence point theorems. We work with capacities that take each component in a set X to a ball Bx in a ball space (Z, B). We initially demonstrate a general theorem that isn't itself an coincidence point theorem, yet entirely adaptable in its applications. The key thought is to utilize a self-assertive attestation P(x) on the components x ∈ X. In Section 6.2 we present two fundamental uses of this general theorem, considering capacities f, g : X → Y . In the principal application, we take X to be a ball space and utilize a capacity from X to its balls, taking the declaration P(x) to state that f(Bx) ∩ g(Bx) 6= ∅. In the second application, we take Y to be a ball space and utilize a capacity from X to the balls in Y, taking the declaration P(x) to state that fx, gx ∈ Bx. In Section 6.3, we present another Bx-type coincidence point theorem for ultrametric spaces which is an exceptional instance of our last application, and we contrast it and the coincidence point theorem of Prieß-

Crampe and Ribenboim and the ultrametric form of Goebel's theorem.

2. Ball Spaces

In this paper we will introduce a generalized completeness property by defining so-called spherically complete ball spaces. We will demonstrate a grouping of ball spaces dependent on their fulfillment properties and delineate this order by models. In the second part we will take a gander at the properties of capacities characterized on ball spaces.

Definition and classification of ball spaces

Definition: A ball space is a pair (X, B), where X is a nonempty set and B is a nonempty collection of nonempty subsets of X. The elements of B will be called balls.

Note that we do not require any topology here. Our definition gives us flexibility and we can adapt it to many cases, for example:

- intervals (closed or open) in an ordered set,
- metric (closed or open) balls in a metric space,
- any kind of ultrametric balls in an ultrametric space,
- closed or open subsets of a topological space

We can also restrict attention to any nonempty subset of a given set of balls. In particular, we will be interested in nests of balls.

Definition: A nest of balls is a nonempty collection of balls which is totally ordered by inclusion. Lemma 3.1.3. Let (X, B) be a ball space. Then the set of all nests which contain a given ball B0 ∈ B has maximal elements.

Proof. Take B0 ∈ B. Let C be the set of all nests which contain B0:

C = {N ⊆ B | N is a nest and B0 ∈ N }.

This set is halfway arranged by consideration. Take any subset F of C with the end goal that (F, ⊆) is a completely requested set. At that point S N ∈F N is a completely requested set, so it is a home containing B0 since B0 ∈ N for all N ∈ C. This home is an upper destined for the chain F ⊆ C. Subsequently by Zorn's Lemma the set C contains no less than one maximal component N0.

The essential culmination property which we will present now, is obtained from ultrametric spaces.

Definition: A ball space (X, B) is called spherically complete if the intersection of every nest of balls is nonempty.

Example: A roundly total ultrametric space together with the group of shut ultrametric balls is a circularly total ball space.

Example: By Theorem 2.0.25, a minimized topological space with the group of all nonempty shut subsets is a roundly total ball space.

Example: The reals, as an arranged set, with the group of all nonempty, shut and limited interims is a roundly total ball space, despite the fact that it isn't smaller under the determined topology. Presently we will see what culmination of a measurement space implies in the dialect of roundly total ball spaces.

Theorem: Take a metric space (X, d) and a set S ⊆ R + which has 0 as its unique limit point. Define a ball space on X as

BS := {Br(x) | x ∈ X, r ∈ S}.

At that point the measurement space (X, d) is finished if and just if the ball space (X, BS) is roundly entire. Verification. First guess that (X, d) is an entire measurement space. Take any home N of shut measurement balls in BS. On the off chance that the home contains a littlest ball, its convergence is nonempty; so we accept that is does not. Since S has 0 as its one of a kind limit point, we have that S is discretely requested, and each unbounded slipping chain in S can be filed by the characteristic numbers. Hence the home N is of the frame

(Brn (xn))n∈N,

where rn > rn+1 for every n ∈ N and

limn→∞ rn = 0

Take > 0 and N ∈ N such that rN < 2 . Since (Brn (xn))n∈N is a nest, we have that the ball BrN (xN) contains xm, xn for every m, n > N. Therefore, d(xm, xn) ≤ 2rN < . We have shown that (xn)n∈N is a Cauchy sequence. Let y be its limit. We have to show that y lies in the intersection of all balls in N . Take a ball Brn (xn) ∈ N and suppose that y /∈ Brn (xn). Then d(xn, y) > rn, and we set ε := d(xn, y) − rn > 0. Since limn→∞ rn = 0, there is m ∈ N such that m > n and d(xm, y) < ε. Since N is a nest, we have that xm ∈ Brn (xn), so d(xn, xm) ≤ rn . Thus,

rn + ε = d(xn, y) ≤ d(xn, xm) + d(xm, y) < rn + ε,

a contradiction. We have demonstrated that y ∈ Brn (xn) for each n ≥ N. Thus, y is in the convergence of the home N, demonstrating that (X, BS) is circularly total. Presently accept that (X, BS) is circularly entire. Take any Cauchy arrangement (xn)n∈N in X. By our suspicions on S, we can pick a succession (si)i∈N in {s ∈ S | s < s0} with the end goal that 0 < 2si+1 ≤ si. By enlistment on I ∈ N we pick an expanding grouping (ni)i∈N of regular numbers with the end goal that the balls Bi := Bsi (xni) frame a home.

3. Functions on Ball Spaces

We will take a closer look at the properties of functions on ball spaces. We will likewise present some extra properties that capacities characterized on a ball space can have. We begin with the accompanying suggestion which indicates when the property of circular fulfillment is safeguarded under a capacity between ball spaces.

Proposition: Let (X1, B1) and (X2, B2) be ball spaces and f : X1 → X2 a function. Suppose that the preimage of every ball in (X2, B2) is a ball in (X1, B1). If N is a nest of balls in (X2, B2), then the preimages of the balls in the nest N form a nest of balls in (X1, B1). If (X1, B1) is spherically complete, then also (X2, B2) is spherically complete.

Proof. Let N be a nest of balls in (X2, B2). Then the preimage of every ball in N is a ball in (X1, B1) by assumption. Since the balls in N are totally ordered by inclusion, their preimages are also totally ordered by inclusion, so they form a nest of balls in B1.

Now assume that (X1, B1) is roundly entire. Take a home N in (X2, B2) and the accumulation N 0 = {f −1 (B) | B ∈ N } of preimages in X1 , which is a home in B1. Note that f(T N 0) ⊆ T B∈N f(f −1 (B)) ⊆ T N . The crossing point of N 0 is nonempty by supposition, so it contains some x ∈ X1. At that point f(x) ∈ T N. This demonstrates (X2, B2) is circularly entire.

Presently we will consider the capacities characterized on a ball space with the qualities in a similar space and characterize a few properties a ball may have concerning such capacities, and in addition properties of the capacities themselves. We will utilize them in the following segments.

Definition: The function f : X → X on a ball space (X, B) is called strongly contracting on orbits if there is a function

X 3 x 7−→ Bx ∈ B

such that the following conditions hold for all x ∈ X:

(S1) x ∈ Bx,
(S2) Bfx ⊆ Bx, and if x 6= fx, then Bf ix ⊂ 6= Bx for some i ≥ 1.

Definition: Take a function f : X → X on a ball space (X, B) which is strongly contracting on orbits. A nest of balls N ⊂ B is called an f-nest if N = {Bx | x ∈ S} for some set S ⊆ X which is closed under f.

Definition: The function f is called self-contractive if it is strongly contracting on orbits and satisfies:

(S3) if N is an f-nest and if z ∈ T N, then Bz ⊆ T N.

4. Fixed point theorems with definition and proof

We say that a function f from a set X into itself has a fixed point if there is x ∈ X such that fx = x. Fixed point theorems try to find proper conditions on the function f and the space X to obtain a fixed point. In this segment we will contemplate the capacity f in two cases. First we will characterize the capacity f on a ball space (X, B) and put a few conditions on balls in B to get the settled point. From that point we will consider the capacity f on a ball space (X, B) to be emphatically contracting on circles and find adequate conditions to guarantee the presence of a settled point. In the two cases, we will apply the general theorems in metric and ultrametric space to demonstrate Banach's Fixed Point Theorem and a theorem of S. Prieß-Crampe and P. Ribenboim.

General fixed point theorems

Theorem: (General Fixed Point Theorem). Take a ball space (X, B) and a function f: X → X such that the following conditions are satisfied:

(GF1) every ball B ∈ B is f-closed (see Definition 3.2.2),

(GF2) every non-singleton ball B ∈ B properly contains some B0 ∈ B,

(GF3) the intersection of every nest of balls in B is a singleton or contains some B ∈ B. Then f has a fixed point in every ball.

Proof. Take any ball B0 ∈ B. Then by Lemma 3.1.3, there is a maximal nest N containing B0. Since every ball in B is f-closed,

f(\ N) ⊆ \ B∈N f(B) ⊆ \ B∈N B = \ N ,

that is, T N is f-closed. By condition (GF3), T N is a singleton or contains a ball B ∈ B. If T N is a singleton, we obtain a fixed point. So assume that N isn't a singleton. At that point it contains a ball B ∈ B. In the event that B isn't a singleton, by (GF2), it appropriately contains a ball B0 ∈ B. Be that as it may, N S {B0} is then a home that legitimately contains N , and this repudiates the maximality of N . Accordingly B must be a singleton {x}, and x is a settled point since B = {x} is f-shut by (GF1). Additionally x ∈ B0 since B is contained in B0 in light of the fact that B0 is a ball in the home N. Thus f has a settled point in each ball in B.

Theorem: Let f be a function on a ball space (X, B) such that the following conditions are satisfied:

(F1) there is at least one f-contracting ball (see Definition 3.2.2) in the ball space,

(F2) for every f-contracting ball B ∈ B there is an f-contracting ball in the image f(B),

(F3) the intersection of every nest of f-contracting balls contains an f-contracting ball. Then f admits a fixed point.

Proof. Define B 0 := {B ∈ B | B is an f-contracting ball}. By (F1) B 0 6= ∅, so (X, B 0) is a ball space. Every f-contracting ball is in particular f-closed, so (GF1) in Theorem 4.1.1 holds (for B 0 in place of B). Take any non-singleton ball B ∈ B0 . Then by (F2) there is an f-contracting ball B0 in the image f(B). Hence B0 ⊆ f(B) ⊂ 6= B which proves that (GF2) holds. Now to show that also (GF3) holds, take a nest of f-contracting balls N. Then by (F3), there is B0 ∈ B0 such that B0 ⊆ T N . Thus (GF3) holds. So now we can apply the General Fixed Point Theorem to (X, B 0) to obtain a fixed point x0 in every ball B ∈ B0 .

In the following subsections we will apply the General Fixed Point Theorem to prove Banach’s Fixed Point Theorem and a fixed point theorem due to Prieß-Crampe and Ribenboim.

Application to metric spaces

Definition: Let (X, d) be a metric space. A function $f : X \to X$ is said to be a contracting if there is a positive real number $c < 1$ such that $d(fx, fy) \le cd(x, y)$ for all $x, y \in X$.

Theorem: (Banach's Fixed Point Theorem). Every contracting function on a complete metric space (X, d) has a unique fixed point.

Proof. Let f be a contracting function on a complete metric space (X, d). Then for every $x \in X$ we have:

$d(fx, f 2x) \le cd(x, fx)$.

Consequently,

$d(x, fix) \le d(x, fx) + d(fx, f 2x) + ... + (f i-1 , fix)$
$\le d(x, fx) + cd(x, fx) + ... + c i-1 d(x, fx)$
$\le d(x, fx)(1 + c + c 2 + ... + c i-1)$
$\le d(x, fx)$ X∞ i=0 c i = $d(x, fx)$ 1 1 – c

where we were able to use the value of the geometric series since $0 < c < 1$. We fix $x \in X$, set $d := d(x, fx)$ and define:

Bx-type fixed point theorems

Theorem: Take a function f on a ball space (X, B) which is strongly contracting on orbits (see Definition). If for every f-nest N (see Definition 3.2.4) in (X, B), there is some z ∈ T N such that Bz ⊆ T N, then f has a fixed point.

Proof. Let f be a function on a ball space (X, B) which is strongly contracting on orbits. Then for every x ∈ X, the set {Bf ix | i ≥ 0} is an f-nest. Henceforth the arrangement of all f-homes is nonempty. It is halfway arranged by consideration since the association over a climbing chain of f-homes is again a f-settle. So by Zorn's Lemma, there is a maximal f-settle N. By the supposition of the theorem, there is some z ∈ T N with the end goal that Bz ⊆ T N. Assume that z 6= fz. At that point by (S2), we have that Bf iz ⊂ 6= Bz for some I ≥ 1. Since Bf iz ⊂ 6= T N for some I ≥ 1, the set N S {Bf kz | k ≥ i} is a f-settle which appropriately contains N . However, this repudiates the maximality of N. Along these lines z is a settled point.

Application to metric spaces

Theorem: (Banach's Fixed Point Theorem). Every contracting function on a complete metric space (X, d) has a unique fixed point.

Proof. Let f be a contracting function on a complete metric space (X, d). Then by inequality, for x ∈ X we have

$d(x, fix) \le d(x, fx)$ 1 − c .

For x ∈ X and i ≥ 0, let

Bf ix : = {y ∈ X | d(y, fix) ≤ c i 1 − c d(x, fx)}.

Consider the ball space (X, B), where B = {Bf ix | i ≥ 0}. We wish to prove that f is self-contractive. By definition of the balls, x ∈ Bx so (S1) holds. To prove that (S2) holds, take any element y ∈ Bfx. Then by the fact that f is contracting, we have:

$d(x, y) \le d(x, fx) + d(fx, y)$
≤ d(x, fx) + c 1 − c d(x, fx) = d(x, fx) 1 − c.
So y ∈ Bx. Therefore Bfx ⊆ Bx.

We have that f ix ∈ Bx for all i ≥ 0 since d(x, fix) ≤ d(x,fx) 1−c . Since c < 1, there is some i ∈ N such that

As we show in the proof of Theorem the set S = { c i 1−c d(x, fx) | i ≥ 0} of radii of the balls Bf ix has 0 as its unique limit point. Since the metric space X is complete, by Theorem 3.1.8 the ball space (X, B) is spherically complete. Take an f-nest N . Then T N 6= ∅. So there is some z ∈ X such that z ∈ T N . We wish to show that Bz ⊆ T N. Take any Bx ∈ N, then Bf ix ∈ N for all i > 0 since N is an f-nest.

5. Conclusion

In this paper we have studied a general Bx type theorem that is not itself a coincidence point theorem, but allows high flexibility in its applications. Further it very well may be derived from the general Bx type theorem coincidence point theorems for two particular cases which depend on either the area or co-space of the capacities under thought being been a ball space. In the last, unique sorts of coincidence point theorems for ultrametric spaces can be presented.

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