Volume-03
Issue-10
October-2018

ISSN: 2455-3085 (Online)
RESEARCH REVIEW International Journal of Multidisciplinary
www.rrjournals.com [UGC Listed Journal]

Spectral decomposition of compact self adjoint operators in infinite dimensional spaces

Indu Ratti

Assistant Professor, Sikh National College, Banga, Punjab (India)

ARTICLE DETAILS

Article History
Published Online: 10 October 2018

Keywords
Eigen, Matrix, adjoint

ABSTRACT

Eigen decomposition or spectral decomposition is the factorization of a matrix into a canonical form, where the matrix is the form of eigen values and eigen vectors. Here we are talking about diagnolizable matrices. Spectral Theorem provides spectral decomposition, Eigen value decomposition of the underlying vector space on which the operator acts. Here, we have tried to work on the formulation of an operator explicitly, operator being self adjoint and compact defined on Hilbert space.

1. Introduction

Decomposition of a self adjoint operator is quite complicated. But it can be done in an better , easier way if we are working with a compact operator. If H be a Hilbert space and T is defined from H to H. The complete spectral decomposition of T can be stated in a quite elementary fashion. Spectral Theorem is a generalization of the familiar theorem from Linear algebra asserting that a self adjoint n x n matrix A can be diagnolized

In particular a compact self adjoint operator can be unitarily diagonalized. Actually, spectral theory is an inclusive term for theories extending the Eigen vector and Eigen value theory of single matrix to a much broader theory of operators in a variety of mathematical spaces.

This paper is concerned with studying the "spectral representation of compact self adjoint operators. Here, we have tried to concentrate on infinite dimensional spaces. Although, the proofs for both the finite and infinite dimensional cases have been discussed in standard books, but we have tried to prove it with slightly different approach. The concept is relatively straight forward for operators on finite dimensional spaces but will require some modifications for operators on infinite dimensional spaces.

For infinite dimensional spaces, we started with family of projections. We represented T in terms of Riemann Steiltzes Integral. Considering the fact that Eigen values of T can both be positive as well as negative. We discussed both the cases and concluded the result. But before going further, we shall require a few facts concerning the terms and definitions used in the Theorem , which are as follows

2. Important Definitions

1. **Eigen Values :**

If $A : X \to Y$ be a linear operator then scalar $\lambda \in K$ (R or C) is called Eigen value of A if $\exists\, x \in X$ $(x \neq 0)$ s.t $Ax = \lambda x$ i.e x is called Eigen vector of A corresponding to Eigen value λ

2. **Self Adjoint Operator :**

A bounded operator A on a Hilbert space H is said to be self adjoint if $A^* = A$, where A^* is adjoint of A.

3. **Kernel of an Operator :**

If $A : X \to Y$ b a Linear operator then

$N(A) = \{ x \in X : Ax = 0\}$ is called kernel of A or null space of A.

4. **Spectral Family :**

A real spectral family (or real decomposition of unity) is a one parameter family $\xi = (E_\lambda)_{\lambda \in R}$ of projections E_λ defined on a Hilbert space H which depends on real parameter λ and is s.t.

a) $\lambda < \mu$ and $E_\lambda \leq E_\mu$ implies $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$

b) Lt $E_\lambda x = 0$ as $\lambda \to -\infty$

c) Lt $E_\lambda x = x$ as $\lambda \to \infty$

5. **Cauchy's Sequence**

A sequence $< x_n >$ in X is said to be a cauchy's sequence if for every $\in > 0 \;\exists\; p \in N$ such that

$\| x_n - x_m \| < \in \;\forall\; n, m \geq p$

3. Statement of Theorem

Let $T : H \to H$ be a compact and self adjoint operator on a Hilbert space H. Then there is a finite or infinite sequence $\{\lambda_n\}_{n=1}^{N}$ ($n \in Z^+$ Or $N = \infty$) of real eigen values $\lambda_n \neq 0$ and a corresponding orthonormal sequence $\{e_n\}_{n=1}^{N}$ in H such that

(a) $Te_n = \lambda_n e_n \;\forall\; n$ with $1 \leq n \leq N$

(b) $N(T) = \text{span}(\{e_n\}_{n=1}^{N})$

(c) if $N = \infty$ then $\lambda_n = 0$ as $n \to \infty$

Before going further, we will prove some very important results for the theorem which are given in the form of lemma:

Lemma1 : Let $T: H \to H$ be a compact and self adjoint operator on a Hilbert space H. Let. $S_1 = S(0,1)$ be unit sphere in H. Then there is a vector $x_0 \in S_1$ such that $\| Tx_0 \| = \| T \|$

Proof : We know that $T = \sup \|Tx\|$ for all $x \in S_1$

$\therefore \exists$ Sequence x_1, x_2 ...... S_1 s.t. $Lt_{n\to\infty} \| Tx_n \| = \| T \|$.........*

Let $Lt\, Tx_n = y_0$ (because T is compact)

Let Y = cl(span { x_1, x_2 ....}), which is closed subspace of H, and a closed subspace of a Hilbert space is Hilbert space in itself.

$\Rightarrow$ Y is reflexive (We know that every Hilbert space is reflexive)

By construction, Y is separable

there is a subsequence of sequence $< x_n >$ which converges weakly in Y. Let us name the subsequence as $< x_{jk} >$ i.e. $(x_{j1}, x_{j2}, x_{j3}....)$ with $1 \le j_1 < j_2 < j_3$................. and this subsequence converges to $x_0 \in Y$

Now, if $z \in Y$

$< Tx_0, z > = < x_0\, T^* z > = < x_o , T_Z > = Lt_{k\to\infty\infty} < x_{jk}, Tz >$

$= Lt < Tx_{jk}, z > = < y_0 , z >$

Here third inequality holds because $< x_{jk} >$ cgs weakly to x_0 and last inequality holds because $Lt\, Tx_{jk} = y_o$ and <.,.> is continuous Concluding $< Tx_0, z > = < y_0, z >$

$Tx_0 = y_o$

Hence $\|Tx_0\| = \|y_0\| = \| LtT(x_n) \| = Lt \|Tx_n\| = \|T\|$ (from *)

i.e $\|Tx_0\| = \| T \|$ which we had to prove.

But the aim of the above Lemma is also to prove that $x_o \in S_1$,

Where $S_1 = S(0,1)$

Now, $|< x_{jk}, x_0 >| \le \| x_{jk} \| \|x_o\| = 1 \quad \forall k$

By Cauchy Schwarz inequality

Here, we can't take $\|x_0\| < 1$ because then

$\|Tx_o\| \le \|T\| \|x_0\| \le \|T\|$ which will be contrary to above proved .Hence $\|x_0\| = 1$ ie $x_o \in S_1$

Lemma 2: Let X be a finite dimensional inner product space and T is a self adjoint operator on X, Then T can be represented as $T = \int_A^B \lambda\, dE_\lambda$

Proof : We know that if T is a self adjoint operator, then it can be presented as $T = \sum_{i=1}^{k} \lambda_i \ P_i$

When $\lambda_1\ \lambda_2 ...... \lambda_k$ are distinct Eigen values of T and for each

$i \in \{ 1, ... k\}$, P_i is the orthogonal projection onto

$N(A - \lambda_i I)$

without loss of generality, Let us assume

$\lambda_1 < \lambda_2 <$ .... $< \lambda_k$

and $E_o = 0,\ E_1 = P_1,\ E_2 = P_1 + P_2, \ldots E_i = P_1 + P_2 + ... + P_i$,

$i = 1, 2 \ldots k$, We can write $T = \sum_{i=1}^{k} \lambda_i \ (E_{i-}\, E_{i-1})$

and $0 = E_o \le E_1 \le E_2 \le$ .... $\le E_k = I$

The representation of T given above can be viewed as a Riemann – Stieltzes integral with respect to a projection value function as follows. For $\lambda \in R$, define

$E(\lambda) =$ $\{0$ if $\lambda < \lambda_1$

$P_1 + P_2 + ... + P_i$ if $\lambda_i \le \lambda < \lambda_{i+1}$, $i = 1, \ldots k-1$

I if $\lambda \ge \lambda_k \}$

Taking, $t_1^{(n)}, t_2^{(n)} .... t_n^{(n)}$ Set $A < \lambda_1 < t_1^{(n)} < .... < t_n^{(n)} = \lambda_k = B$

s.t $\max \{ t_j^{(n)} - t_{j-1} = 1, ...n \} \to 0$ as $n \to \infty$

We can write $T = \sum_{i=1}^{k} \lambda_i \ [E(\lambda_i) - E(\lambda_{i-1})]$

$= Lt_{n\to\infty}\, \beta_j^{(n)} [E(t_j^{(n)}) - E(t_{j-1}^{(n)})]$

Where $\beta_1^{(n)}, ... \beta_n^{(n)}$ satisfy $t_{j-1}^{(n)} < \beta_j^{(n)} \le t_j^{(n)}$, $j = 1, \ldots n$

Then T can be written as Riemann Stieltzes Integral in the form $T = \int_A^\beta \lambda d\, E(\lambda)$

The proof can be modeled on one of the standard methods of proofs in the finite dimensional case, although the details are of course more difficult in infinite dimension.

Proof of the Theorem:

Let T : H → H be compact self adjoint operator defined on Hilbert space H. By using Lemma2, We can represent T in terms of Riemann Integral as

$T = \int_m^M \lambda dE\lambda$ Where m = Inf $< Tx,, x >$ for $\| x \| = 1$

M = Sup $< Tx, x >$ for $\| x \| = 1$

For taking value $\lambda = m$ into consideration, we take

$T = \int_{m-0}^{M} \lambda dE\lambda$

If we consider two real members $A < 0 < B$ s.t $A < m$ and $B > M$ then $T = \int_A^B \lambda dE_\lambda$ .... (1)

Let $Y_\lambda = E_\lambda (H)$ which is closed subspace of H

and E_λ is projection of H onto Y_λ

Now, we will try to prove that dim $Y_{\lambda o}^{\perp}$ is finite for any $\lambda_0 > 0$

Let us suppose the contrary i.e Let dim $Y_{\lambda 0}^{\perp} = \infty$

Now, we can have two options : Either $\lambda \le \lambda_0$ or $\lambda \ge \lambda_0$

Firstly if $\lambda \le \lambda_0$ then $E_\lambda\, E_{\lambda 0} = E_\lambda$

and if $\lambda \ge \lambda_0$ then $E_\lambda\, E_{\lambda 0} = E_{\lambda 0}$ (Definition of spectral family) Multiplying both sides of (1) by $E_{\lambda 0}$

i.e. $T \quad E_{\lambda 0} = T = \left(\int_A^B \lambda dE_\lambda \right) E_{\lambda 0} = \int_A^\lambda \lambda dE_\lambda$ .... (2)

($\therefore$ Here $\lambda \le \lambda_0 \Rightarrow E_\lambda E_{\lambda 0} = E_\lambda$)

Substract (2) from (1)

$T - TE_{\lambda 0} = \int_{\lambda 0}^{B} \lambda dE_\lambda$

$T(I - E_{\lambda 0}) = \int_{\lambda 0}^{B} \lambda dE_\lambda \ \ge \ \int_{\lambda 0}^{B} \lambda_0 dE_\lambda = \lambda_0 [E_B - E_{\lambda 0}] = = \lambda_0\ I - E_{\lambda 0})$

($\therefore$ Here range is λ_0 to B) $\therefore T(I - E_{\lambda 0}) \ge \lambda_0 (I - E_{\lambda 0})$ .... (3)

Here $E_{\lambda 0}$ is projection of H onto $Y_{\lambda 0}$

$\Rightarrow I - E_{\lambda 0}$ is projection of H onto $Y_{\lambda 0}^{\perp}$

$\Rightarrow$ If $y \in Y_{\lambda 0}^{\perp} \Rightarrow (I - E_{\lambda 0})\, y = y$

By using (3) and partial order relation

$< Ty, y > \ge < \lambda_0 y, y > = \lambda_0 \| y \|^2$

Rewriting, $\lambda_0 \| y \|^2 = < Ty, y > \le \|Ty\| \|y\|$

By Cauchy's Schwarz Inequality $, \Rightarrow \lambda_0 \| y \| \le \|Ty\|$

i.e. $\| Ty \| \ge \lambda_0 \| y \| \quad \forall y \in Y_{\lambda 0}^{\perp}$

But we have supposed that dim $Y_{\lambda 0}^{\perp} = \infty$

$\therefore$ There is an infinite orthonormal sequence $e_1, e_2 \ldots$ in $Y_{\lambda 0}^{\perp}$

s.t $\forall j \ne i$, $|| Te_j - Te_i || \ge \lambda_0 \ || e_j - e_i|| = \sqrt{2}\, \lambda_0$

This shows that there can't exist any subsequence $j_1 < j_2 <$.....

s.t $< Te_j >$ is cauchy's sequence .

But $<e_j>$ is bounded sequence in H and T is given to be compact. ∴ Every bounded sequence should have a convergent subsequence. But corresponding to $<e_j>$, $<Te_j>$ is not convergent subsequence. This is contradiction to the fact that T is compact**.** Hence, our supposition is wrong .

∴ dim $Y_{\lambda 0}{}^{\perp}$ is finite for each $\lambda_0 > 0$ and Let dim $Y_{\lambda 0}{}^{\perp} = d(\lambda)$

By def. of spectral family, d (λ) is decreasing sequence.

∴ for each $n \in Z^+$, the set $\{\lambda > 0 : d(\lambda) = n\}$ is either empty or an interval of the form $R^+ \cap (\mu, \mu')$

It means that there is an infinite sequence $B = \mu, > \mu_2 >$ .... of positive members s.t d (λ) is constant on each interval $[\mu_j, \mu_{j-1})$ and Lt $\mu_j = 0$as $j \to \infty$

By using definition of Riemann steiltzes integral, we claim that $\int_0^B \lambda dE_\lambda = \sum_{j=1}^{\infty} \mu_j\ (E\mu_j - E\mu_{j+1})$

Proof for claim : Let $P_n = \{0, \mu_n, \mu_{n-1}, \ldots \mu_1 = B\}$ be partition of [0, B] Here $0 < \mu_n < \mu_{n-1} \ldots < \mu_2 < \mu_1 = B$

Again splitting subinterval (μ_j, μ_{j-1}) , $2 \le j \le n$ into n equal parts

By construction, $\|P_n\| \to 0$ as $n \to \infty$

By definition Lt S $(P_n) = \int_0^B \lambda dE_\lambda$ as $n \to \infty$ .... *

Now $S(P_n) = \sum_{J=1}^{\infty} \mu_j (E\mu_j - E\mu_{j+1})$

$= \mu_n (E\mu_n - E\mu_{n+1}) + \sum_{j=1}^{n-1} \mu_j (E\mu_j - E\mu_{j+1})$

$= \mu_n (E\mu_n - E_0) + \sum_{J=1}^{n-1} \mu_j (E\mu_j - E\mu_{j+1})$

Hence, $\| S(P_n) - \sum_{J=1}^{n-1} \mu_j (E\mu_j - E\mu_{j+1}) \| = \| \mu_n (E\mu_n - E_0) \|$

$\le \mu_n \to 0$ as $n \to \infty$(∴$E\mu_n - E_0$ is projection and its norm ≤ 1)

∴ By def. of Limit $Lt_{n\to\infty} S(P_n) = \sum_{J=1}^{n-1} \mu_j (E\mu_j - E\mu_{j+1})$……. **

i.e. $Lt_{n\to\infty} S(P_n) = \sum_{J=1}^{\infty} \mu_j (E\mu_j - E\mu_{j+1})$

combining * and **

$$\int_0^B \lambda dE_\lambda = \sum_{j=1}^{\infty} \mu_j (E\mu_j - E\mu_{j+1}) \quad \ldots\ldots\ (4)$$

i.e our claim is established

Now, in the next section, we will try to represent the above integral in terms of members of some orthonormal sequence. Consider the projection $E\mu_j - E\mu_{j+1}$, which is projection onto $Y\mu_{j+1}{}^{\perp} \cap Y\mu_j$

Let us name as $Z_j = Y\mu j_{+1}{}^{\perp} \cap Y\mu j$

Now, we will prove that for $j \ge 1$, all z_1, z_2 ... are mutually orthogonal . Now for $\forall\ 1 \le j < i$

$\Rightarrow \mu_{j+1} \ge \mu_i$ and $\mu_1 > \mu_2 > \mu_3 > \ldots\ldots$ is a decreasing sequence

$\Rightarrow Y\mu i \subset Y\mu_{j+1}$ .... *

and $Z_j = Y\mu_{j+1}{}^{\perp} \cap Y\mu j \quad \Rightarrow Z_j \perp Y\mu_{j+1}$

For j=i, $Z_i = Y\mu_{i+1} \cap Y\mu_i \Rightarrow Z_i \subset Y\mu_i$ .... **

Again * and ** $Z_i \subset Y\mu_i \subset Y\mu_{j+1}$ and $Z_j \perp Y\mu_{j+1}$

∴ $Z_i \subset Y\mu_{j+1}$ and $Z_j \perp Y\mu_{j+1}$

$\Rightarrow Z_i \perp Z_j\ \forall\ i, j$

∴ Z_1, Z_2, ... are mutually orthogonal. We can choose an ON basis $e_1, e_2 \ldots e_{d1}$ in Z_1 , then similarly choose an ON basis in Z_2 and so on, we obtain an orthonormal sequence e_1, e_2 ... in H, which may be finite or infinite.

∴ we can write $(E\mu_j - E\mu_{j+1})x = x = \sum_{ek \in zj} <x, e_k> e_k$ for each j

From (4) ,$\int_0^B \lambda dE_\lambda = \sum_{f=1}^{\infty} \mu_j (E\mu_j - E\mu_{j+1})$

$\left(\int_0^B \lambda dE_\lambda\right) = \sum_{j=1}^{\infty} \mu_j (E\mu_j - E\mu_{j+1})x$

$= \sum_k \lambda_k <x, e_k> \forall\ x \in H$ (here $\lambda_k := \mu_j$)

i.e $\left(\int_0^B \lambda dE_\lambda\right) = \sum_k \lambda_k <x, e_k> e_k$

Here, by construction $\lambda_1 \ge \lambda_2 \ge \ldots..$ ie it is a decreasing sequence of positive members and if this sequence is infinite Lt $\lambda_k = 0$ as $k \to \infty$ Because $E\mu_j - E\mu_{j+1}$ is projection onto Z_j and $e_1, e_2 \ldots$ ..is ON in H.

Here $Z_j = Y^{\perp}\mu_{j+1} \cap Y\mu_j$

$\Rightarrow Z_j \subset Y\mu_{j+1}{}^{\perp}$ and $Y_o \subset Y\mu_{j+1}$ ic $Y_0{}^{\perp} \supset Y\mu_{j+1}{}^{\perp}$

combing $Z_j \subset Y^{\perp}\mu_{j+1} \subset Y_o{}^{\perp}$

$Z_j \subset Y_o{}^{\perp}\ \forall\ j$

ie All $e_1, e_2, \ldots.. \in Z_j \subset Y_o{}^{\perp}$

Now, we will take the case when $\lambda_0 < 0$.Firstly, we will prove dim $Y_{\lambda 0} < \infty$

Let us start with $T = \int_A^B \lambda dE_\lambda$

$TE_{\lambda 0} = \int_A^B \lambda dE_\lambda E_{\lambda\ 0}$

$= \int_A^{\lambda 0} \lambda dE_\lambda$ [Because $\lambda \le \lambda_0\ E_\lambda E_{\lambda 0} = E_\lambda$

$\le \int_A^{\lambda 0} \lambda dE_\lambda = \lambda_0 [E_{\lambda 0} - E_A] = \lambda_0 E_{\lambda 0}$

That is $TE_{\lambda 0} \le \lambda_0 E_{\lambda 0}$

Let $y \in Y_{\lambda 0}$

Now $< Ty, y> \le < \lambda_0 y, y> = \lambda_0 \|y\|^2 < 0$ (∴$\lambda_0 < 0$) ..(5)

Again $| < Ty, y > | \le \|Ty\| \|y\|$ ……………(6)

(5) and (6) $\lambda_0 \|y\|^2 \le \|Ty\| \|y\|$

ie $\lambda_0 \|y\| \le \|Ty\|$

ie $\|Ty\| \ge \lambda_0 \|y\|\ \ \forall\ y \in Y_{\lambda 0}$, but we suppose dim $Y_{\lambda 0} = \infty$ ∴ There is an infinite orthonormal sequence f_1, f_2 ... in $Y_{\lambda 0}$ such that

$\|T_{fj} - T_{fi}\| \ge \lambda_0 \|f_j - f_i\| = \sqrt{2}\ \lambda_0$

This shows that there can’t exist any subsequence such that $< T_{fj} >$ is cauchy’s sequence. But $< f_j>$ in bounded sequence in H and T is given to be compact.

∴Every bounded sequence should have a convergent subsequence. But corresponding to $< f_j >$, $< T_{fj}>$ is not convergent subsequence. This is contradiction to the fact that T is compact. Hence our supposition is wrong.

∴ dim $Y_{\lambda 0}$ is finite for $\lambda_0 < 0$. and the set of all these λ’s is either empty or an interval of the form $[\nu, \nu')$. It follows that there is an infinite sequence $A = \nu_1 < \nu_2 < \ldots.$ of negative numbers such that Lt $\nu_j = 0$ as $j \to \infty$

Now, by using definition of Riemann Steiltzes Integral, we claim that $\int_A^0 \lambda dE_\lambda = \sum_{j=2}^{\infty} \nu_j\ [E\nu_j - E\nu_{j-1}]$

Let $p'_n = \{ A = \nu_1, \nu_2 \ldots. \nu_n = 0 \}$ be partition of [A, 0]

By the construction as before, $\|P'_n\| \to 0$ as $n \to \infty$

By definition Lt $_{n\to\infty}S(P'_n) = \int_A^0 \lambda dE_\lambda$ ……………… (7)

Now $S(P'_n) = \sum_{j=2}^{\infty} \nu_j\, [E\nu_j - E\nu_{j-1}]$

$S(P'_n) = \nu_n [E\nu_n - E\nu_{n-1}] + \sum_{j=2}^{n-1} \nu_j\, [E\nu_j - E\nu_{j-1}]$

$S(P'_n) = \nu_n [E_0 - E\nu_{n-1}] + \sum_{j=2}^{n-1} \nu_j\, [E\nu_j - E\nu_{j-1}]$

$\| S(P'_n) - \sum_{j=2}^{n-1} \nu_j\, [E\nu_j - E\nu_{j-1}] \| = \| \nu_n [E_0 - E\nu_n] \| \le \nu_n \to 0$

as $n \to \infty$ (since $E_o - E\nu_n$ is projection)

∴ By def. of limit, Lt $_{n\to\infty} S(P'_n) = \sum_{j=2}^{n-1} \nu_j\, [E\nu_j - E\nu_{j-1}]$

ie. Lt $_{n\to\infty} S(P'_n) = \sum_{j=2}^{\infty} \nu_j\, [E\nu_j - E\nu_{j-1}]$ …………… (8)

Combining (7) and (8)

$\int_A^0 \lambda dE_\lambda = \sum_{j=2}^{\infty} \nu_j\, [E\nu_j - E\nu_{j-1}]$

Now, in the next section, we will try to represent the above integral in terms of members of some orthonormal sequence.

Now, $E\nu_j - E\nu_{j-1}$ is projection of H onto $Y^{\perp}\nu_{j+1} \cap Y\nu_j$

let us name $Q_j = Y\nu_{j+1}{}^{\perp} \cap Y\nu_j$

Now, we will prove that $j \ge 2$, All Q_1, Q_2 ... are mutually orthogonal.

For $2 \le j < i$,

$\Rightarrow Y\nu i \subset Y\nu_{j-1}$

Now $Q_i = Y\nu_{i-1}{}^{\perp} \cap Y\nu_I$,It neans $Qi \subset Y\nu_i \subset Y\nu_{j-1}$

ic $Qi \subset Y\nu_{j-1}$ and because $Q_j \perp Y\nu_{j-1}$

combing $Q_i \perp Q_j \;\forall\; i, j$

∴ Q_1, Q_2 ... are mutually orthogonal.

∴ We can choose an orthonormal basis f_1, f_2 ... in H (by combining all ON basis in Q_1, Q_2...) which may be finite or infinite.

we can write $[E\nu_j - E\nu_{j-1}]\, x = x = \sum_{x\in H} < x, f_k > f_k$ for each k.

Now, considering $\int_A^0 \lambda dE_\lambda = \sum_{j=2}^{\infty} \nu_j\, [E\nu_j - E\nu_{j-1}]x$

$$\left(\int_A^0 \lambda dE_\lambda\right)x = \sum_{j=2}^{\infty} \nu_j\, [E\nu_j - E\nu_{j-1}]x$$

$$\left(\int_A^0 \lambda dE_\lambda\right)x = \sum_k \lambda_k' < x, f_k > f_k, \text{ here } \nu_j := \lambda'_k$$

Now, concluding the above discussion, we have derived the following results i.e.

$\left(\int_A^0 \lambda dE_\lambda\right)x = \sum_k \lambda_k' < x, f_k > f_k$

and $\left(\int_0^B \lambda dE_\lambda\right)x = \sum_k \lambda_k < x, e_k > e_k$

Combining , $Tx = \left(\int_A^B \lambda dE_\lambda\right)x = \left(\int_A^0 \lambda dE_\lambda\right)x + \left(\int_0^B \lambda dE_\lambda\right)x$

$= \sum_k \lambda_k < x, e_k > e_k + \sum_k \lambda_k' < x, f_k > f_k$

Here $e_k \perp f_k \;\forall\; k$. and $e_k \in Y_o^{\perp}$ and $f_k \in Y_o$

∴ Any merging of the sequence $<e_k>$ and $<f_k>$ is a new orthonormal sequence. Hence, it follows $Te_k = \lambda_k e_k \;\forall\; e_k$

and $Te_k = \lambda_k f_k \;\forall\; f_k$ which is last part of theorem.

For the second part, we have to prove *N* (T) = span $(\{en\}_{n=1}^{\infty})^{\perp}$

If $x \in N(T) \Rightarrow Tx = 0$

$\Rightarrow < x, e_k > = 0$ and $<x, f_k> = 0$

$\Rightarrow x \perp e_k$ and $x \perp f_k$

$\Rightarrow x \in \text{span}(\{e_k\} \cup \{f_k\})^{\perp}$

Hence $N(T) \subset \text{span}(\{e_k\} \cup \{f_k\})^{\perp}$

and similarly $\text{span}(\{e_k\} \cup \{f_k\})^{\perp} \subset N(T)$

combining , $N(T) = \text{span}(\{e_k\} \cup \{f_k\})^{\perp}$ which is required result.

Hence, all claims of theorem are fulfilled

4. Conclusion

In this paper, we basically discussed the spectral representation of compact self adjoint operators. The significance and usefulness of this result lies in the fact that we can represent T (ν),ν ∈ H in a simple and unique form. So, we are writing H as, $H = Y + Y^{\perp}$, where we assume $Y = \text{span}(\{e_n\}_{n=1}^{N})$

We then proved kernel of T as orthogonal of the set which we proved as combination of the orthonormal elements.

This process is done when N is infinite. This made study of infinite dimensional spaces both richer as well as crucial. In the proof, we expressed T in terms of projections and then using some very important results ,we presented T in terms of Riemann Steiltzes Integral. That Integral was written in terms of some orthonormal sequence for required result. Motivated for the application of this theorem, we here tried to write $v \in H$ in the form $\nu = (\sum \alpha_n e_n) + Z$,

Where $\alpha_n \in k$, $Z \in N(T)$ and for each such vector ν,we have

$$T(\nu) = \sum \lambda_n \alpha_n e_n$$

which is required representation and this presentation is unique.

References

1. Andreas Strombergsson, Spectral Theorem for Compact Self Adjoint Operators(preprint).
2. Kreyszig, Introductory Functional Analysis With Applications, Wiley Classics Library Edition, 1989.
3. Nair, M.T., Functional Analysis: A First Course, Prentice Hall of India Private Limited, New Delhi,2008.