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Colorings and zero-divisor graphs in commutative rings

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ABSTRACT

The zero-divisor graphs have appealed to algebraists and graph theorists alike by establishing an interplay between ring theory and graph theory. The techniques used in studying zero divisor graphs have varied from simple computational to highly refined ring theoretic. Rings with special type of zero –divisor graphs have been classified in literature. In this article, we give a brief survey of zero-divisor graphs in commutative rings.

1. Zero divisor graphs in commutative rings

Beck in 1988 established a relation between ring theory and graph theory by introducing the idea of colorings of a commutative ring. He expressed the hope that the interplay will turn out to be fruitful for both the branches of Mathematics. In his seminal work, Beck mainly studied the finite colorable rings. For a commutative ring R, he introduced a graph whose vertices are elements of the ring R, and the two vertices x, y are adjoined by an edge if and only if $xy = 0$ in the ring.

For a ring $R, \chi(R)$ denotes the chromatic number of the graph associated with the ring R as introduced by Beck. Recall that chromatic number of the graph is the minimal number of colors that can be assigned to the vertices of the graph so that no two adjacent vertices have the same color. A subset $C = \{x_1, x_2, x_3, \dots\}$ is called a *clique* if $x_i x_j = 0$ for all $i \neq j$. If in a ring R, every clique has at most n elements and there is a clique with n elements, then Beck defined the *clique number of R* to be n, written as *cl*$(R) = n$ but if there is no upper bounds on the sizes of the cliques in R, then *clique* R is defined to be ∞. It is clear that $\chi(R) \geq$ *cl* (R). But for a large classes of rings like reduced rings and principal ideal rings, Beck observed that $\chi(R) =$ *cl* (R). Motivated by this, Beck conjectured that for any ring R, $\chi(R) =$ *cl* (R) even though there were examples available for general graphs for which chromatic number of graph was strictly bigger than clique number of the graph. But Anderson and Naseer [6] provided a counterexample for the conjecture. They constructed a finite local ring with *cl*$(R) = 5$ and $\chi(R) = 6$.

It is a well known result in Graph theory that a graph is 2-colorable if and only if does not contain any odd cycle. Using this result, Beck proved that $\chi(R)$ and *cl*(R) are equal for rings with smaller chromatic number. He proved the following result:

Theorem 1.1 (Beck [15]) *Let R be a commutative ring for which $\chi(R)$ is finite. k be an integer ≤ 4, then $\chi(R) = k$ if and only if cl$(R) = k$. Moreover, $\chi(R) = 5$ implies cl$(R) = 5$.*

Beck also gave a complete characterization for commutative rings with $\chi(R) \leq 3$ as follows:

Theorem 1.2 (Beck [15]) (1*)* $\chi(R) = 1$ *iff* $R = \{0\}$.

(2) $\chi(R) = 2$ iff R is a finite field, or $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[x]/(x^2)$.

(3) $\chi(R) = 3$ iff R is isomorphic to one of the following rings:

- *(a) A product of two finite fields.*
- *(b) $k \times \mathbb{Z}_4$, k a finite field.*
- *(c) $k \times \mathbb{Z}_2[x]/(x^2)$, k a finite field.*
- *(d)* $\mathbb{Z}_8$.
- *(e)* $\mathbb{Z}_9$.
- *(f)* $\mathbb{Z}_3[x]/(x^2)$.
- *(g)* $\mathbb{Z}_2[x]/(x^3)$.
- *(h)* $\mathbb{Z}_4[x]/(2x, x^2 - 2)$.

Beck also proved that chromatic number and clique number of ring of integers modulo any n are also equal:

Theorem 1.3 (Beck [15]) $\chi(\mathbb{Z}_n) =$*cl*$(\mathbb{Z}_n) = p_1^{n_1} \dots p_k^{n_k} q_1^{m_1} \dots q_r^{m_r} + r$*, where* $n = p_1^{2n_1} \dots p_k^{2n_k} q_1^{2m_1+1} \dots q_r^{2m_r+1}$ *such that* $p_1, \dots, p_k, q_1, \dots, q_r$ *are also distinct primes.*

Beck proved that if R is a commutative reduced ring such that $\chi(R) < \infty$, then R has only a finite number of minimal prime ideals and if n is this number, then $\chi(R) =$ *cl* $(R) = n + 1$. He also proved the following result:

Theorem 1.4 (Beck [15]) (1) *If R is a commutative reduced ring such that $\chi(R) < \infty$, then $\chi(I) =$ cl(I) for any ideal $I \subset R$.*

(2) If R is a commutative principal ideal ring with $\chi(R) < \infty$, then $\chi(I) =$ cl(I) for any ideal $I \subset R$.

Beck also characterized the rings R for which $\chi(R) < \infty$ as in the following:

Theorem 1.5 (Beck [15]) *For a commutative ring, the following are equivalent:*

- *(1)* $\chi(R)$ *is finite.*

(2) *cl* (R) *is finite.*
(3) *The nilradical in* R *is finite and equals a finite intersection of prime ideals.*
(4) R *does not contain an infinite clique.*

A commutative ring whose chromatic number is finite or which satisfies any of the equivalent conditions of the previous theorem was called a *Coloring* by Beck. Beck proved that any commutative Coloring has an a.c.c on ideals of the form *Ann* a, where *Ann* $a = \{x \in R : ax = 0\}$. He proved that subring of a Coloring is also a Coloring and finite product of Colorings is again a Coloring. He also proved that in a Coloring, the set of associated primes is finite, that is the set of prime ideals that are of the form *Ann* x, is finite. Further, he proved that a minimal prime of a Coloring R is an associated prime of R and if R is an associated prime of R, then either R_P is a field and hence P is minimal or R/P is a field and hence P is maximal. In other words, any associated prime ideal in a commutative Coloring is either a minimal ideal or a maximal ideal.

Anderson and Naeser in [6] called a commutative ring R a *chromatic ring* if $\chi(R) =$ *cl*(R). They showed that the ring $R = \mathbb{Z}_4[x,y,z]/(x^2-2, y^2-2, z^2, 2x, 2y, 2z, xy, xz, yz-2)$ is not a chromatic ring and $\chi(R) = 6$ and *cl*$(R) = 5$. In this ring, $J^3 = 0$ but $J^2 \neq 0$, where J is nilradical of R. For commutative ring with $J^2 = 0$, they proved the following:

Theorem 1.6 (Anderson and Naseer [6]) *Let* R *be a commuttaive Coloring with nilradical* J *such that* $J^2 = 0$, *then* R *is a chromatic ring with* $\chi(R) =$ *cl*$(R) = |\,J\,| + \varepsilon(R)$, *where* $\varepsilon(R)$ *denotes the number of minimal prime ideals* P *for which* R_P *is a field.*

For commutative Noetherian rings, the following was proved by Anderson and Naseer in [6]:

Theorem 1.7 (Anderson and Naseer [6]) *Let* R *be a commutative Noetherian ring. Then* R *is a Coloring if and only if* R *is a subring of a finite direct product of fields and a finite ring.*

Complete characterization of structure of commutative rings with $\chi(R) \leq 3$ was given by Beck in [15]. Anderson and Naseer gave complete characterization of structure of rings with $\chi(R) = 4$.

Theorem 1.8 (Anderson and Naseer [6]) *Let* R *be a finite commutative ring with* $\chi(R) = 4$. *Then* R *is isomorphic to one of the following* 31 *types of rings (where* k_i *is a finite field):*

(1) $\mathbb{Z}_4 \times \mathbb{Z}_4$ *(2)* $\mathbb{Z}_4 \times \mathbb{Z}_2[x]/(x^2)$
(3) $\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2[x]/(x^2)$ *(4)* $k_1 \times k_2 \times k_3$
(5) $k_1 \times k_2 \times \mathbb{Z}_4$ *(6)* $k_1 \times k_2 \times \mathbb{Z}_2[x]/(x^2)$ *(7)* $k_1 \times \mathbb{Z}_8$
(8) $k_1 \times \mathbb{Z}_9$ *(9)* $k_1 \times \mathbb{Z}_3[x]/(x^2)$ *(10)* $k_1 \times \mathbb{Z}_2[x]/(x^3)$
(11) $k_1 \times \mathbb{Z}_4[x]/(2x, x^2-2)$ (12) $\mathbb{Z}_{16}$ (13) $\mathbb{Z}_2[x]/(x^4)$
(14) $\mathbb{Z}_4[x]/(x^2+2x+2)$ *(15)* $GF(2^2)[x]/(x^2)$
(16) $\mathbb{Z}_4[x]/(x^2+x+1)$ *(17)* $\mathbb{Z}_2[x,y]/(x,y)^2$
(18) $\mathbb{Z}_4[x]/(2.x)^2$ *(19)* $\mathbb{Z}_{27}$ *(20)* $\mathbb{Z}_3[x]/(x^3)$
(21) $\mathbb{Z}_9\,[x]/(x^2-6, 3x)$ *(22)* $\mathbb{Z}_2[x,y]/(x^2, y^2-xy)$
(23) $\mathbb{Z}_4[x]/(2x, x^3-2)$ *(24)* $\mathbb{Z}_4[x]/(x^2-2)$
(25) $\mathbb{Z}_9\,[x]/(x^2-3, 3x)$ *(26)* $\mathbb{Z}_2[x,y]/(x^2, y^2)$
(27) $\mathbb{Z}_8[x]/(2x-4, x^2)$ *(28)* $\mathbb{Z}_4[x]/(x^2)$
(29)) $\mathbb{Z}_4[x]/(x^2-2x)$ *(30)* $\mathbb{Z}_4[x,y]/(x^2, xy-2, y^2, 2x, 2y)$
(31) $\mathbb{Z}_4[x,y]/(x^2, xy-2, y^2-xy, 2x, 2y)$.

Beck and Anderson et al. had let all the elements of the commutative ring under consideration as vertices in their graph and they mainly focused their attention to the colorings in their graphs associated with the rings. But later, Anderson and Livingston changed this approach. They only considered nonzero zero divisors of the ring as the vertices in the corresponding graph instead of all the elements of the ring. If R is a commutative ring with 1 and $Z(R)$ denotes the set of all zero-divisors of R, and $Z(R)^* = Z(R) \setminus \{0\}$, then zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple graph whose vertices are elements of $Z(R)^*$ and for distinct elements $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. It is clear that for a commutative ring R, if the graph associated with R as studied by Beck and Anderson et al. is denoted by $\Gamma_0(R)$, then $\Gamma_0(R)$, can be considered as a subgraph of $\Gamma(R)$. Anderson and Livingston observed the following:

Theorem 1.9 (Anderson and Livingston [4]) *Let* R *be a commutative ring. Then* $\Gamma(R)$ *is finite if and only if either* R *is finite or an integral domain. In particular, if* $1 \leq |\Gamma(R)| < \infty$, *then* R *is finite and not a field.*

Recall that a graph $\Gamma$is called *connected* if there is a path between any two distinct vertices. For distinct vertices x, y of Γ, let $d(x,y)$ denote the length of the shortest path from x to y and if there is no path joining x to y, then $d(x,y)$ is defined to be ∞. *Diameter of a graph* Γ, denoted by *diam* $(\Gamma) =$ *Sup* $\{\, d(x,y) : x, y$ *are distinct vertices of* $\Gamma\}$. The *girth of graph* Γ, denoted by $g(\Gamma)$ is defined as the length of the shortest cycle in Γ. $g(\Gamma)$ is defined to be ∞ if there are no cycles in Γ. Anderson and Livingston proved the following result regarding diameter and girth of the zero-divisor graph $\Gamma(R)$ associated with a commutative ring R.

Theorem 1.10 (Anderson and Livingston [4]) *Let* R *be a commutative ring. Then* $\Gamma(R)$ *is connected and diam* $\Gamma(R) \leq 3$. *Moreover, if* $\Gamma(R)$ *conatins a cycle, then* $g\big(\Gamma(R)\big) \leq 7$.

Later, Mulay [23] improved the upper bound for girth of the zero-divisor graph of a commutative ring, while DeMeyer [19] gave an easier proof for the result:

Theorem 1.11 (Mulay [23]) *Let* R *be a commutative ring, if* $\Gamma(R)$ *contains a cycle, then* $g\big(\Gamma(R)\big) \leq 4$.

Anderson and Livingston also characterized those commutative rings R, for which there is a vertex in $\Gamma(R)$, which is adjacent to every other vertex:

Theorem 1.11 (Anderson and Livingston [4]) *Let* R *be a commutative ring. Then there is a vertex of* $\Gamma(R)$, *which is adjacent to every other vertex if and only if either* $R \cong \mathbb{Z}_2 \times A$, *where* A *is an integral domain or* $Z(R)$ *is an annihilator ideal.*

Recall that a graph Γ is called complete if any two distinct vertices are adjacent. It is clear that for a commutative ring

R, $\Gamma(R)$ is complete if and only if $xy = 0$ for all distinct $x, y \in Z(R)$. Anderson and Livingston showed that except for the case when $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we also have $x^2 = 0$, for all $x \in Z(R)$ whenever $\Gamma(R)$ is complete.

Theorem 1.12 (Anderson and Livingston [4]) *Let R be a commutative ring, then $\Gamma(R)$ is complete if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$.*

Theorem 1.13 (Anderson and Livingston [4]) *Let R be a finite commutative ring. If $\Gamma(R)$ is complete, then either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is a local ring with*
char $R = p$ or p^2 and $|\Gamma(R)| = p^n - 1$, where p is a prime and $n \geq 1$.

Conversely, Anderson and Livingston were able to construct a commutative ring R for each prime p and $n \geq 1$ such that $\Gamma(R)$ is complete and $|\Gamma(R)| = p^n - 1$. Thus K^m, the complete graph on m vertices can be realized as $\Gamma(R)$ for some commutative ring if and only if $m = p^n - 1$ for some prime p and integer $n \geq 1$.

Recall that a *complete bipartite graph* is a graph whose vertex set can be partitioned into two disjoint subsets U, V such that each vertex in U is adjacent to each vertex in V and every edge in the graph connects a vertex in U to a vertex in V. If the first set U has m elements and the second set V has n elements then the corresponding complete bipartite graph is denoted by $K^{m,n}$. A complete bipartite graph of the type $K^{1,n}$ is called a *star graph.* Regarding, the zero-divisor graphs being star graphs, the following is known:

Theorem 1.14 (Anderson and Livingston [4]) *Let R be a finite commutative ring with $|\Gamma(R)| \geq 4$, then $\Gamma(R)$ is a star graph if and only if $R \cong \mathbb{Z}_2 \times F$, where F is a finite field. In particular, if $\Gamma(R)$ is a star graph then $|\Gamma(R)| = p^n$ for some prime p, and integer $n \geq 0$. Conversely, for any prime p and non-negative integer n, the star graph of order p^n can be realized as $\Gamma(R)$.*

From Theorem 1.11, it follows that girth of a zero-divisor graph of any commutative ring is $3, 4$ or ∞. Anderson and Mulay in [23] gave complete characterizations for each case as the following results:

Theorem 1.15 (Mulay [23]) *The following statements are equivalent for a commutative ring R with nil(R) nonzero:*
1. *(1) gr$(\Gamma(R)) = 4$.*
2. *(2) $R \cong D \times B$, where D is an integral domain with $|D| \geq 3$ and $B \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.*
3. *(3) $\Gamma(R) = \overline{K}^{m,3}$ with $m \geq 2$. (Here $\overline{K}^{m,3}$ is the graph formed by joining the complete bipartite graph $G_1 = K^{m,3}$ $(= A \cup B$ with $|A| = m, |B| = 3)$ to the star graph $G_2 = K^{1,m}$ by identifying center of G_2 and a point of B.)*

Theorem 1.16 (Mulay [23]) *The following statements are equivalent for a reduced commutative ring R:*
- *(1) $\Gamma(R)$ is nonempty with gr$(\Gamma(R)) = \infty$.*
- *(2) $T(R) = \mathbb{Z}_2 \times K$, where K is a field and $T(R)$ is the total quotients ring of R.*
- *(3) $\Gamma(R) = K^{1,n}$ for some $n \geq 1$.*

When the ring is not reduced, characterizations are as follows:

Theorem 1.17 (Mulay [23]) *The following statements are equivalent for a commutative ring R with nil (R) nonzero:*
- *(1) gr $(\Gamma(R) = \infty$.*
- *(2) $R \cong B$ or $\mathbb{Z}_2 \times B$, where $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$, or $\Gamma(R)$ is a star graph.*
- *(3) $\Gamma(R)$ is a singleton, a $\overline{K}^{1,3}$ or a $K^{1,n}$ for some $n \geq 1$.*

Axtell et al. in [10] observed that for $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathit{\Gamma}(R)$ is complete but $\mathit{\Gamma}(R[x])$ and $\mathit{\Gamma}(R[[x]])$ are complete bi-partite graphs and hence diam $\mathit{\Gamma}(R[x]) = \mathit{\Gamma}\big(R[[x]]\big) = 2$. But the following result holds for the rings that are not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$:

Theorem 1.18 (Axtell et al. [10]) *If $R \neq \mathbb{Z}_2 \times \mathbb{Z}_2$ is a commutative ring, then the following are equivalent:*
- *(1) $\mathit{\Gamma}(R)$ is complete.*
- *(2) $\mathit{\Gamma}(R[x])$ is complete.*
- *(3) $\mathit{\Gamma}(R[[x]])$ is complete.*

Axtell et al. observed that in general, girth of $\mathit{\Gamma}(R)$ and girth of $\mathit{\Gamma}(R[x])$ may not be equal. For instance, it can be seen that for $R = \mathbb{Z}_4$, gr$(\big(\mathit{\Gamma}(R)\big) = \infty$, but gr $(\mathit{\Gamma}(R[x]) = 3$. But situation is much better if zero divisor graph of R contains a cycle.

Theorem 1.19 (Axtell et al. [10]) *Let R be a commutative ring not necessarily with identity. Then $gr\,\big(\Gamma(R)\big) \geq gr\,\big(\mathit{\Gamma}(R[x])\big) = gr\,(\mathit{\Gamma}\big(R[[x]]\big)$. In addition, if R has no nilpotents and $\mathit{\Gamma}(R)$ contains a cycle, then $gr\big(\mathit{\Gamma}(R)\big) = gr\,\big(\mathit{\Gamma}(R[x])\big) = gr\,\Big(\mathit{\Gamma}\big(R[[x]]\big)\Big)$.*

Using above result by Axtell et al., Anderson and Mulay gave the following complete information about the girth of polynomial rings and power series rings over a commutative rings:

Theorem 1.20 (Anderson and Mulay [5]) *Let R be a commutative ring.*
- *(1) Suppose that $\Gamma(R)$ is nonempty with gr $\Gamma(R)) = \infty$.*
  - *(a) If R is reduced, then*
    gr $\big(\mathit{\Gamma}(R[x])\big) =$ gr $\Big(\mathit{\Gamma}\big(R[[x]]\big)\Big) = 4$.
  - *(b) If R is not reduced, then*
    gr $\big(\mathit{\Gamma}(R[x])\big) =$ gr $\Big(\mathit{\Gamma}\big(R[[x]]\big)\Big) = 3$.
- *(2) If gr$(\Gamma(R)) = 3$, then*
  gr $\big(\mathit{\Gamma}(R[x])\big) =$ gr $\Big(\mathit{\Gamma}\big(R[[x]]\big)\Big) = 3$.
- *(3) If gr$\big(\Gamma(R)\big) = 4$, then*
  - *(a) If R is reduced, then*
    gr $\big(\mathit{\Gamma}(R[x])\big) =$ gr $\Big(\mathit{\Gamma}\big(R[[x]]\big)\Big) = 4$.
  - *(b) If R is not reduced, then*
    gr $\big(\mathit{\Gamma}(R[x])\big) =$ gr $\Big(\mathit{\Gamma}\big(R[[x]]\big)\Big) = 3$.

The characterizations for diameters of the zero divisor graphs of polynomial rings and power series rings over commutative rings are bit tedious and were discovered by Lucas in [21]. In the following, a *McCoy ring* is meant to be a ring R whose any finitely generated ideal contained in $Z(R)$ has a nonzero annihilator.

Theorem 1.21 (Lucas [21]) *Let R be a reduced ring that is not an integral domain. Then $1 \le$ diam $\Gamma(R) \le$ diam $\Gamma(R[x]) \le$ diam $\Gamma\left(R[[x]]\right) \le 3$. Moreover, all the possible sequences for three numbers are as follows:*

1. *diam $\Gamma(R) = 1$ and diam $\Gamma(R[x]) =$ diam $\Gamma\left(R[[x]]\right) = 2$ if and only if R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.*
2. *diam $\Gamma(R) =$ diam $\Gamma(R[x]) =$ diam $\Gamma\left(R[[x]]\right) = 2$ if and only if either R has exactly two minimal primes and is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or for each pair of countably generated ideals I and J with nonzero annihilators, the sum $I + J$ has a nonzero annihilator (and R is a McCoy ring with $Z(R)$ an ideal).*
3. *diam $\Gamma(R) =$ diam $\Gamma(R[x]) = 2$ and diam $\Gamma(R[[x]])$ =3 if and only if R is a McCoy ring with $Z(R)$ an ideal but there exists countably generated ideals I and J with nonzero annihilators such that $I + J$ does not have a nonzero annihilator.*
4. *diam $\Gamma(R) = 2$ and diam $\Gamma(R[x]) =$ diam $\Gamma(R[[x]]) = 3$ if and only if $Z(R)$ is an ideal and each two generated ideal contained in $Z(R)$ has a nonzero annihilator but R is not a McCoy ring.*
5. *diam $\Gamma(R) =$ diam $\Gamma(R[x]) =$ diam $\Gamma(R[[x]]) = 3$ if and only if R has more than two minimal primes and there is a pair of zero divisors a, b such that the ideal generated by a and b does not have a nonzero annihilator.*

An ideal of a ring R is called *divided* if it is comparable with each principal ideal of R. For non-reduced rings, the following results on the diameter of power series ring were proved by Lucas:

Theorem 1.22 (Lucas [21]) *Let R be a nonreduced ring with prime nilradical Nil(R). If Nil (R) is divided and $Z(R)$ properly contains Nil (R), then diam $\Gamma\left(R[[x]]\right) = 3$.*

Theorem 1.23 (Lucas [21]) *Let R be a nonreduced ring R such that $Z(R)$ is not the nilradical of R. If $Z(R)$ has a nonzero annihilator, then $Z(R)$ is an ideal of R, R is a McCoy ring, diam $\Gamma(R)$ =diam $\Gamma(R[x]) = 2$ and $Z(R)[[x]] \subseteq Z\left(R[[x]]\right)$. Moreover, if $Z(R)Nil\ (R) = (0)$, then diam $\left(\Gamma\left(R[[x]]\right) = 2$.*

It is clear that if two ring R and S are isomorphic then their zero divisor graphs $\Gamma(R)$ and $\Gamma(S)$ are also isomorphic. But on the other hand, the zero divisor graphs of the rings $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $S = \mathbb{Z}_9$ are isomorphic but the rings are clearly not isomorphic. In this direction, the following results are known:

Theorem 1.24 (Anderson et al. [1]) *Let R and S be finite reduced commutative rings which are not fields, then $\Gamma(R) \cong \Gamma(S)$ if and only if $R \cong S$.*

This result has been generalized in Akbari and Mohammadian in [7]:

Theorem 1.25 (Akbari and Mohammadian [7]) *Let R be a finite reduced commutative ring and S not an integral domain. If $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$, unless $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_6$ and S is a local ring.*

Theorem 1.26 (Anderson et al. [2]) *Let $\{R_i\}_{i\in I}$ ($|I| \ge 2$) and $\{S_j\}_{j\in J}$ be two families of integral domains, and let $R = \prod_{i\in I} R_i$ and $S = \prod_{j\in J} S_j$. Then $\Gamma(R) \cong \Gamma(S)$ if and only if there is a bijection $\varphi: I \longrightarrow J$ such that $|R_i| = |S_{\varphi(i)}|$ for each $i \in I$. In particular, if $\Gamma(R) \cong \Gamma(S)$ and each R_i is a finite field, then each S_j is also a finite field and $R_i \cong S_{\varphi(i)}$ for each $i \in I$, and thus $R \cong S$.*

The following theorem by Anderson et al. [2] suggests that one would perhaps not be able to characterize many classes of commutative rings solely in terms of their zero divisor graphs:

Theorem 1.27 (Anderson et al. [2]) *Let R be a commutative ring with total quotient ring $T(R)$, then the graphs $\Gamma(R)$ and $\Gamma(T(R))$ are isomorphic.*

From Theorem 1.26, it follows that the graphs of the rings $R = \mathbb{Z}_2 \times \mathbb{R}$ and $S = \mathbb{Z}_2 \times \mathbb{C}$ have isomorphic zero-divisor graphs but clearly the rings are not isomorphic implying that von Neumann regular rings are not determined by their zero divisor graphs. But on the other hand, following result by LaGrange shows that Boolean rings are determined by their zero divisor graphs:

Theorem 1.28 (LaGrange [22]) *Let R be a ring with nonzero zero-divisors, and not isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[x]/(x^2)$. If S is a Boolean ring, such that $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$. In particular, if R and S are Boolean rings, then $\Gamma(R) \cong \Gamma(S)$ if and only if $R \cong S$*

A graph is called *planar* if it can be embedded (that is, can be drawn with no crossings) in the plane. Result due to Kuratowski (see [18]) tells that a graph G is planar if and only if it contains no subdivisions homeomorphic to K^5 or $K^{3,3}$. A finite commutative ring is the direct product of a finitely many local rings, so to characterize finite commutative rings whose zero-divisor graphs are planar, one needs to get a bound on the number of local factors that can be used and then the case of local rings needs to be handled separately. If $\Gamma(R)$ is planar, then Akbari et al. [9] observed that R can have at most three local ring factors, as otherwise the graph would contain a $K^{3,3}$. Anderson et al. proved the following result:

Theorem 1.29 (Anderson et al. [2]) (a) *Let $R = \mathbb{Z}_n$, where $n \ge 2$ is not prime. Then $\Gamma(R)$ is planar if and only if $n \in \{8, 12, 16, 18, 25, 27\} \cup \{2p, 3p | p\ is\ a\ prime\}$.*

(b) Let $R = \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r}$, where $r \ge 2$ and $2 \le n_1 \le \cdots \le n_r$. Then $\Gamma(R)$ is planar if and only if R is one of $\mathbb{Z}_2 \times \mathbb{Z}_{4,}$ $\mathbb{Z}_2 \times$

$\mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_p$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_9$, $\mathbb{Z}_3 \times \mathbb{Z}_q$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, *where* p *and* q *are primes.*

Theorem 1.30 (Anderson et al. [2]) *Let* $R_{n,m} = \mathbb{Z}_n[x]/(x^m)$, *where* $m, n \geq 2$.

(a) $\Gamma(R_{n,2})$ *is planar if and only if* $n \leq 5$.

(b) $\Gamma(R_{n,3})$ *is planar if and only if* $n \leq 3$.

(c) $\Gamma(R_{n,4})$ *is planar if and only if* $n = 5$.

(d) $\Gamma(R_{n,m})$ *is never planar if* $m \geq 5$.

Akbari et al. proved the following result for local rings:

Theorem 1.31 (Akbari et al. [9]) *Let* (R, M) *be a finite commutative local ring. Then* $\Gamma(R)$ *is not planar if one of the following holds:*

(a) $|R/M| \geq 4$, *and* $|R| \geq 26$.

(b) $|R/M| = 3$ *and* $|R| \geq 28$.

(c) $|R/M| = 2$ *and* $|R| \geq 33$.

In this article, we have only focused on zero-divisor graphs of commutative rings with unity. The idea has been extended to non-commutative rings also, where there are several possible definitions and graphs considered can be directed or undirected. The idea has also been extended to other algebraic structures like semigroups and modules over commutative rings.

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