

# A Model with Quantity Discount, Pricing and Partial Backordering for a Deteriorating Inventory

<sup>1</sup>Devendra Singh & <sup>2</sup>Dinesh Singh

<sup>1</sup>Assistant Prof. Haryana Institute of Public Administration, Gurgaon (India)

<sup>2</sup>Associate Prof., S.G.R.R. (P.G.) College, Dehradun (India)

## ARTICLE DETAILS

### Article History

Published Online: 07 September 2018

### Keywords

Inventory, Discount, Pricing

## ABSTRACT

Inventory model with quantity discount, pricing and partial backordering, although common in practice, have received very little attention from researchers. The objective of this study is to develop a deterministic inventory model with quantity discount, pricing and partial backordering when the product in stock deteriorates with time. Most retailers make pricing decisions of their products at certain times of the year and these decisions affect demand. In this study a power law form of the price dependence of demand is considered. The rate of deterioration is taken to be linear function of time. The model is solved analytically. A numerical example is given to illustrate the theory.

## 1. Introduction

Deteriorating inventory, in general, is defined as decay, damage, spoilage, evaporation, obsolescence, pilferage, loss of marginal value of a commodity that results in decreasing usefulness from the original one. Vegetables, gasoline and semiconductor chips are a few examples of such products. In this study, an algorithm is developed to determine the replenishment and the pricing policy of products when the quantity in stock diminishes in time linearly. The cost of raw material is assumed to have quantity discounts, and demand is deterministic and a function of price. It is assumed that lead times are zero. When there is no shortage, some of the customers will seek alternative supply, while others will wait for the next replenishment. This behavior is demonstrated by the use of partial backordering in the model formulation.

A consideration of partial backordering inventory for non-perishable product has been undertaken by Rosenberg 1979, Kalro and Gohil 1982, and Wee 1989. Inventory theory of deteriorating items has been studied by several authors in recent decades. Ghare and Schrader 1963 have developed an Economic Order Quantity (EOQ) model by assuming a constant rate of deterioration. Covert and Philip 1974, Shah 1976, Dave 1979 and Aggarwal 1979 have assumed instantaneous replenishment rate with different assumptions on the patterns of deterioration, while Elsayed and Teresi 1983 and Rafaat et al. 1991 and Wee 1993 have assumed finite replenishment rate.

Eilon and Mallaya 1966 were the first authors to analyze the pricing policy for perishable item, but they assumed products having a maximum shelf life with no deterioration before the expiration date. An analysis of on-going deteriorating items with price function was done by Cohen 1977. He assumed instantaneous replenishment, and was concerned with simultaneous price setting and ordering levels for exponentially decaying items. Kang and Kim 1983 extended Cohen's model to consider the case for finite

replenishment. Aggarwal and Jaggi 1989 pointed out an approximation error in Cohen and simultaneously found the optimal replenishment level and price for an exponentially decaying product. Abad 1988 considered pricing and lot-size decisions with incremental quantity discounts and partial backordering under a prescribed scheduling period. However, there is a discrepancy in his total cost function where lost sale penalty cost is left out. Recent study considering joint pricing and replenishment policy for inventory with a constant rate of deteriorations developed by Wee 1995. This research, using a more general form of deteriorating patterns, seeks to develop an optimal inventory model with quantity discount, pricing and partial backordering

## 2. Notations

Following notations are used throughout the chapter

- (i) T Constant prescribed scheduling period or cycle length (time units);
  - (ii)  $T_1$ : Duration of inventory cycle when there is positive inventory;
  - (iii) q: The order quantity (units/ cycle);
  - (iv)  $I_m$ : Maximum Inventory level at the start of a cycle (units);
  - (v)  $I_b$ : Back- Ordered quantity at the end of a cycle (units);
  - (vi)  $I_1$ : Lost sale quantity in a cycle (units);
  - (vii)  $d(p)$ : Deterministic demand rate for the product, a function of  $p$  (units/unit time);
  - (viii) B: The fraction of stock out demand back ordered ( $0 < B \leq 1$ );
  - (ix)  $v_i(q_i)$ : Material cost per unit (\$) as a function of  $q_i$ ,  $i$  is the number of price break;
  - (x)  $p$ : Unit selling price (\$) ,  $p > v_i(q_i)$ ;
  - (xi)  $C_1$ : Fixed cost per order (\$/order);
  - (xii)  $C_2(q_i)$ : Holding cost per unit time (\$/unit/unit time);  
 $= C_h + I v_i(q_i)$
- (3.2.1)

where

$C_h$  stands for the holding cost that is independent of the unit cost  $v_i(q_i)$ , while  $I$  is the annual holding cost fraction that is dependent on unit cost  $v_i(q_i)$ ;

(xiii) C3: Shortage cost per unit back ordered per unit time (\$/unit/unit time);

(xiv) C4: Penalty cost of a lost sale including loss of profit(\$/unit);

### 3. Assumptions

In developing the mathematical model of the inventory system, the fundamental assumptions of the model are as follows:

1. The Demand rate is exponential function of selling price. i.e.  $d(p) = ae^{-\beta p}$ , (3.3.1)
2. The replenishment rate is infinite
3. The deterioration rate is linear function of time i.e.
4.  $R(t) = \alpha + \beta t$ , (3.3.2)
5. The replenishment quantity, inventory level, deterioration and demand are treated as continuous variables while the cycle length  $T$  is treated as a discrete variable.
6. The lead time is negligible
7. The replenishment quantity and cycle length are constant for each cycle.
8. Demand during the stock out period is partially lost due to impatient customers.
9. There is no replenishment or repair of deteriorated items during a given cycle.
10. Units are available for satisfying backordered demand immediately after each replenishment.
11. The cost of a deteriorating unit is constant and equal to  $v_i(q_i)$ . This will account for the salvage

value, if any. All- units quantity discount is assumed, the material cost is defined as.

$$v_i(q_i) = \begin{cases} v_1 & \text{for } m_1 < q_1 \leq m_2 \\ v_2 & \text{for } m_2 < q_2 \leq m_3 \\ v_n & \text{for } m_n < q_n \end{cases} \quad (3.3.3)$$

The discounted prices obey the relationships  $v_1 > v_2 > \dots > v_n$ , and  $m_1, m_2, \dots, m_n$  stands for boundaries of the incremental quantities at state 1 to  $n$ .

The objective of this study is to maximize the average total net profit  $NP(T_1, p, T)$ . Next, it is straightforward to calculate the order quantity  $q$ . The scheduling period,  $T$ , can be divided two sub periods; the one with positive on hand inventory during  $T_1$  and the other with shortage during  $(T - T_1)$ . Shortages are partially backordered during the next replenishment.

### 4. Model Development

The Inventory system during a given cycle is depicted in Fig.1. at  $t=0$ , an initial replenishment of  $q$  units is made, of which  $I_b$  units are delivered towards backorders, leaving a balance of  $I_m$  units in the initial inventory. From  $t=0$  to  $t=T_1$  time units, the inventory level decrease due to both demand and deterioration. At  $T_1$ , the inventory level is zero. During the time  $(T - T_1)$  part of the shortage is backlogged and part of it is lost sales. Only the backlogged items are replaced by the next replenishment. The distribution of the time that an item is to determine is linear on  $t$ . i.e the rate of deterioration is  $\alpha + \beta t$ , where  $\alpha, \beta > 0$  and  $0 < \beta < 1$

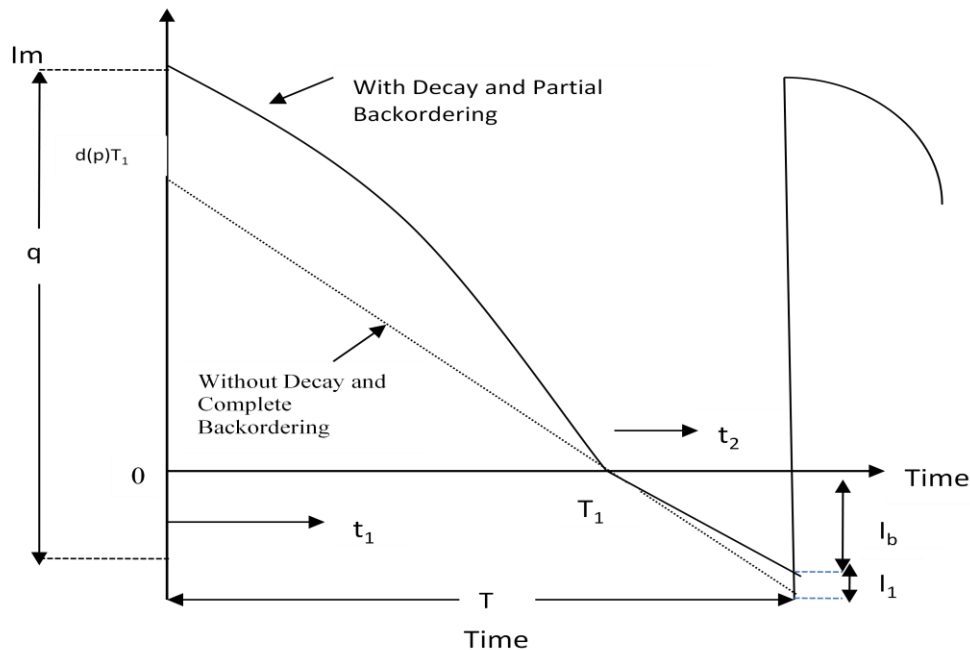


Fig-1

Let  $I(t)$  be the inventory level of the system at time  $t$  ( $0 \leq t \leq T$ ), the differential equation describing  $I(t)$  over the cycle length  $T$  is given as follows:

$$\frac{d}{dt} I(t) + R(t)I(t) = -d(p), \quad 0 \leq t \leq T_1 \quad (3.4.1)$$

and

$$\frac{d}{dt} I(t) = -B \cdot d(p), \quad T_1 \leq t_2 \leq T - T_1 \tag{3.4.2}$$

With initial condition  $I(0) = I_m$  and boundary condition  $I(T_1) = 0$   
 Equation (3.4.1) and (3.4.2) may be written as

$$\frac{d}{dt} I(t_1) + (\alpha + \beta t_1)I(t_1) = -ae^{-bp}, \quad 0 \leq t_1 \leq T_1 \tag{3.4.3}$$

and

$$\frac{d}{dt} I(t_2) = -B \cdot ae^{-bp}, \quad T_1 \leq t_2 \leq T - T_1 \tag{3.4.4}$$

Where  $\alpha \geq 0, 0 < \beta < 1$        $a, b > 0, a > b$

The solutions of differential equations (3.4.3) and (3.4.4) under given condition are respectively given by

$$I(t_1) = ae^{-bp} e^{-(\alpha t_1 + \frac{\beta}{2} t_1^2)} \int_{t_1}^{T_1} e^{(\alpha t + \frac{\beta}{2} t^2)} dt, \quad 0 \leq t_1 \leq T_1 \tag{3.4.5}$$

and

$$I(t_2) = -B \cdot ae^{-bp} (T_1 - t_2), \quad 0 \leq t_2 \leq T - T_1 \tag{3.4.5}$$

Applying the Taylor's series expansions of  $e^{-(\alpha t_1 + \frac{\beta}{2} t_1^2)}$  and  $e^{(\alpha t + \frac{\beta}{2} t^2)}$ , after simplification, one has

$$I(t_1) \approx ae^{-bp} \left[ \left( T_1 + \frac{\alpha}{2} T_1^2 + \frac{(\alpha^2 + \beta)}{6} T_1^3 + \frac{\alpha\beta}{8} T_1^4 + \frac{\beta^2}{40} T_1^5 \right) - \left( 1 + \alpha T_1 + \frac{\alpha^2}{2} T_1^2 + \frac{\alpha\beta}{6} T_1^3 \right) t_1 + \frac{1}{12} (6\alpha + 6(\alpha^2 - \beta)T_1 - 3\alpha\beta T_1^2 - \beta^2 T_1^3) t_1^2 \right. \\ \left. + \frac{1}{2} (-\alpha^2 + \alpha\beta T_1) t_1^3 + (-5\alpha\beta + 3\beta^2 T_1) \frac{1}{24} t_1^4 - \frac{\beta^2}{15} t_1^5 \right] \tag{3.4.6}$$

In particular

$$I_m \approx I(0) = ae^{-bp} \left[ T_1 + \frac{\alpha}{2} T_1^2 + \frac{(\alpha^2 + \beta)}{6} T_1^3 + \frac{\alpha\beta}{8} T_1^4 + \frac{\beta^2}{40} T_1^5 \right] \tag{3.4.7}$$

The Total inventory per cycle  $I_T(T_1)$  is approximately

$$I_T(T_1) \approx \int_0^{T_1} I(t_1) dt_1 \\ = ae^{-bp} \left[ \frac{1}{2} T_1^2 - \frac{\alpha}{6} T_1^3 - \frac{\alpha^2}{24} T_1^4 + \frac{\alpha\beta}{24} T_1^5 + \frac{\beta^2}{90} T_1^6 \right]$$

(3.4.7)

The loss of Stock per cycle due to deterioration,  $D(T_1)$  is

$$D(T_1) = I_m - ae^{-bp} \cdot T_1$$

Total deterioration per cycle is approximately

$$D(T_1) \approx ae^{-bp} \left[ \frac{\alpha}{2} T_1^2 + \frac{(\alpha^2 + \beta)}{6} T_1^3 + \frac{\alpha\beta}{8} T_1^4 + \frac{\beta^2}{40} T_1^5 \right]$$

(3.4.8)

The total shortage backordered per cycle,  $I_b$ , is

$$I_b = B \int_0^{T-T_1} ae^{-bp} t_2 dt_2 = \frac{B}{2} \cdot ae^{-bp} (T - T_1)^2$$

(3.4.9)

The total shortage lost sale,  $I_1$  is

$$I_1 = (1 - B) \int_0^{T-T_1} ae^{-bp} t_2 dt_2 = \frac{(1-B)}{2} ae^{-bp} (T - T_1)^2$$

(3.4.10)

The lot size  $q$  is demand minus lost sale plus deterioration thus

$$q = d(p) \cdot T - \frac{(1 - B)}{2} ae^{-bp} (T - T_1)^2 + D(T_1) \\ = ae^{-bp} \left\{ T - \frac{(1-B)}{2} (T^2 - 2T_1 T + T_1^2) + \frac{\alpha}{2} T_1^2 + \frac{(\alpha^2 + \beta)}{6} T_1^3 + \frac{\alpha\beta}{8} T_1^4 + \frac{\beta^2}{40} T_1^5 \right\} \tag{3.4.11}$$

### 5. Total Cost

The total variable cost per unit time  $K(T_1, p, T)$  is the sum of the material cost (including deterioration cost), the carrying cost, the lost sale penalty cost and the replenishment cost. Thus

$$K(T_1, p, T) = \frac{v_1(q)}{T} + \frac{C_2(q)}{T} I_T(T_1) + \frac{C_3}{T} I_b + \frac{C_4}{T} I_1 + \frac{C_1}{T}$$

$$\begin{aligned}
 &= \frac{1}{T} \left\{ v_i(q) \left\{ T - \frac{(1-B)}{2} (T^2 - 2T_1T + T_1^2) + \frac{\alpha}{2} T_1^2 + \frac{(\alpha^2 + \beta)}{6} T_1^3 + \frac{\alpha\beta}{8} T_1^4 + \frac{\beta^2}{40} T_1^5 \right\} \right. \\
 &\quad + (C_h + Iv_i(q)) \left( \frac{1}{2} T_1^2 - \frac{\alpha}{6} T_1^3 - \frac{\alpha^2}{24} T_1^4 + \frac{\alpha\beta}{24} T_1^5 + \frac{\beta^2}{90} T_1^6 \right) + C_3 \frac{B}{2} (T^2 - 2T_1T + T_1^2) \\
 &\quad \left. + C_4 \frac{(1-B)}{2} (T^2 - 2T_1T + T_1^2) \right\} a e^{-bp} + \frac{C_1}{T}
 \end{aligned} \tag{3.5.1}$$

For the condition with no shortage,  $T_1 = T$ , constant demand,  $d(p) = a$ , and no quantity discount  $v_i=v$ ; equation (3.5.1) becomes

$$K(T) = av \left\{ 1 + \frac{\alpha}{2} T + \frac{(\alpha^2 + \beta)}{6} T^2 + \frac{\alpha\beta}{8} T^3 + \frac{\beta^2}{40} T^4 \right\} + a(C_h + Iv(q)) \left( \frac{T}{2} - \frac{\alpha}{6} T^2 - \frac{\alpha^2}{24} T^3 + \frac{\alpha\beta}{24} T^4 + \frac{\beta^2}{90} T^5 \right) + \frac{C_1}{T} \tag{3.5.2}$$

The total profit per unit time, NP ( $T_1, p, T$ ) is defined as total average revenue minus Total average variable cost. The NP( $T_1, p, T$ ) is given by

$$\begin{aligned}
 NP(T_1, p, T) &= ae^{-bp} \left[ p - \frac{1}{T} \left\{ v_i(q) \left\{ T - \frac{(1-B)}{2} (T^2 - 2T_1T + T_1^2) + \frac{\alpha}{2} T_1^2 + \frac{(\alpha^2 + \beta)}{6} T_1^3 + \frac{\alpha\beta}{8} T_1^4 + \frac{\beta^2}{40} T_1^5 \right\} \right. \right. \\
 &\quad \left. \left. + (C_h + Iv_i(q)) \left( \frac{1}{2} T_1^2 - \frac{\alpha}{6} T_1^3 - \frac{\alpha^2}{24} T_1^4 + \frac{\alpha\beta}{24} T_1^5 + \frac{\beta^2}{90} T_1^6 \right) + \frac{(BC_3 + (1-B)C_4)}{2} (T^2 - 2T_1T + T_1^2) \right\} \right] - \frac{C_1}{T}
 \end{aligned} \tag{3.5.3}$$

With a given value of T, the Total average profit NP ( $T_1, p$ ), can be optimized by taking the partial derivations of NP( $T_1, p$ ) with respect to  $T_1$  and p and setting the derivatives equal to zero.

(3.5.3)  $\implies$

$$\begin{aligned}
 \frac{\partial}{\partial T_1} (NP) &= -\frac{ae^{-bp}}{T} \left\{ v_i(q) \left\{ (1-B)(T - T_1) + \alpha T_1 + \frac{(\alpha^2 + \beta)}{2} T_1^2 + \frac{\alpha\beta}{2} T_1^3 + \frac{\beta^2}{8} T_1^4 \right\} \right. \\
 &\quad \left. + (C_h + Iv_i(q)) \left( T_1 - \frac{\alpha}{2} T_1^2 - \frac{\alpha^2}{6} T_1^3 + \frac{5}{24} \alpha\beta T_1^4 + \frac{\beta^2}{15} T_1^5 \right) - (BC_3 + (1-B)C_4)(T - T_1) \right\}
 \end{aligned} \tag{3.5.4}$$

and

$$\begin{aligned}
 \frac{\partial}{\partial p} (NP) &= ae^{-bp} \left[ 1 - b \left[ p - \frac{1}{T} \left\{ v_i(q) \left\{ T - \frac{(1-B)}{2} (T - T_1)^2 + \frac{\alpha}{2} T_1^2 + \frac{(\alpha^2 + \beta)}{6} T_1^3 + \frac{\alpha\beta}{8} T_1^4 + \frac{\beta^2}{40} T_1^5 \right\} \right. \right. \right. \\
 &\quad \left. \left. + (C_h + Iv_i(q)) \left( \frac{1}{2} T_1^2 - \frac{\alpha}{6} T_1^3 - \frac{\alpha^2}{24} T_1^4 + \frac{\alpha\beta}{24} T_1^5 + \frac{\beta^2}{90} T_1^6 \right) + \frac{(BC_3 + (1-B)C_4)}{2} (T - T_1)^2 \right\} \right] \right]
 \end{aligned} \tag{3.5.5}$$

$$\begin{aligned}
 \frac{\partial^2}{\partial T_1^2} (NP) &= -\frac{ae^{-bp}}{T} \left[ v_i(q) \left\{ -(1-B) + \alpha + (\alpha^2 + \beta)T_1 + \frac{3}{2} \alpha\beta T_1^2 + \frac{\beta^2}{2} T_1^3 \right\} + (C_h + Iv_i(q)) \left( 1 - \alpha T_1 - \frac{\alpha^2}{2} T_1^2 + \frac{5}{6} \alpha\beta T_1^3 + \frac{\beta^2}{3} T_1^4 \right) \right. \\
 &\quad \left. + (BC_3 + (1-B)C_4) \right]
 \end{aligned} \tag{3.5.6}$$

$$\begin{aligned}
 \frac{\partial^2}{\partial p \partial T_1} (NP) &= \frac{abe^{-bp}}{T} \left\{ v_i(q) \left\{ (1-B)(T - T_1) + \alpha T_1 + \frac{(\alpha^2 + \beta)}{2} T_1^2 + \frac{\alpha\beta}{2} T_1^3 + \frac{\beta^2}{8} T_1^4 \right\} \right. \\
 &\quad + (C_h + Iv_i(q)) \left( T_1 - \frac{\alpha}{2} T_1^2 - \frac{\alpha^2}{6} T_1^3 + \frac{5}{24} \alpha\beta T_1^4 + \frac{\beta^2}{15} T_1^5 \right) \\
 &\quad \left. - (BC_3 + (1-B)C_4)(T - T_1) \right\}
 \end{aligned} \tag{3.5.7}$$

and

$$\frac{\partial^2}{\partial p^2} (NP) = -abe^{-bp} \left[ 2 - bp + \frac{b}{T} \left\{ v_i(q) \left( T - \frac{(1-B)}{2} (T - T_1)^2 + \frac{\alpha}{2} T_1^2 + \frac{(\alpha^2 + \beta)}{6} T_1^3 + \frac{\alpha\beta}{8} T_1^4 + \frac{\beta^2}{40} T_1^5 \right) + (C_h + I v_i(q)) \left( \frac{1}{2} T_1^2 - \frac{\alpha}{6} T_1^3 - \frac{\alpha^2}{24} T_1^4 + \frac{\alpha\beta}{24} T_1^5 + \frac{\beta^2}{90} T_1^6 \right) + \frac{(BC_3 + (1-B)C_4)}{2} (T - T_1)^2 \right\} \right] \tag{3.5.8}$$

$$\frac{\partial}{\partial T_1} (NP) = 0 \quad \frac{\partial}{\partial p} (NP) = 0 \text{ and imply}$$

$$v_i(q) \left\{ (1-B)(T - T_1) + \alpha T_1 + \frac{(\alpha^2 + \beta)}{2} T_1^2 + \frac{\alpha\beta}{2} T_1^3 + \frac{\beta^2}{8} T_1^4 \right\} + (C_h + I v_i(q)) \left( T_1 - \frac{\alpha}{2} T_1^2 - \frac{\alpha^2}{6} T_1^3 + \frac{5}{24} \alpha\beta T_1^4 + \frac{\beta^2}{15} T_1^5 \right) - (BC_3 + (1-B)C_4)(T - T_1) = 0 \tag{3.5.9}$$

and

$$1 - bp + \frac{b}{T} \left[ v_i(q) \left\{ T - \frac{(1-B)}{2} (T - T_1)^2 + \frac{\alpha}{2} T_1^2 + \frac{(\alpha^2 + \beta)}{6} T_1^3 + \frac{\alpha\beta}{8} T_1^4 + \frac{\beta^2}{40} T_1^5 \right\} + (C_h + I v_i(q)) \left( \frac{1}{2} T_1^2 - \frac{\alpha}{6} T_1^3 - \frac{\alpha^2}{24} T_1^4 + \frac{\alpha\beta}{24} T_1^5 + \frac{\beta^2}{90} T_1^6 \right) + \frac{(BC_3 + (1-B)C_4)}{2} (T - T_1)^2 \right] = 0 \tag{3.5.10}$$

Equations (3.5.9) and (3.5.10) can be solved simultaneously to derive the values of  $T_1^*$  and  $p^*$  corresponding to any assigned values of parameters (including integral values of  $T$ ) used in the equations with the help of suitable software like **MATLAB**. Since the scheduling period,  $T$ , is usually a discrete variables, the necessary condition for the total average profit  $NP$  to be maximum at  $T_1 = T_1^*, p = p^*, T = T^*$  is

$$NP(T_1^*, p^*, T^*-1) \leq NP(T_1^*, p^*, T^*) \geq NP(T_1^*, p^*, T^*+1) \tag{3.5.11}$$

The value  $NP(T_1^*, p^*, T^*)$  will be maximum if the Hessian conditions are satisfied i.e.

$$\frac{\partial^2}{\partial T_1^2} (NP) < 0, \quad \frac{\partial^2}{\partial p^2} (NP) < 0 \tag{3.5.12}$$

and

$$\left( \frac{\partial^2}{\partial T_1 \partial p} (NP) \right)^2 - \left( \frac{\partial^2}{\partial T_1^2} (NP) \right) \cdot \left( \frac{\partial^2}{\partial p^2} (NP) \right) < 0 \tag{3.5.13}$$

at  $T = T_1^*, p = p^*$  and  $T = T^*$

The numerical example is presented to illustrate the theory.

### 6. Illustrative Example

The following parameters are used to illustrate the theory.

Unit material cost  $v_i(q_i)$  is given by

$$v_i(q_i) = \begin{cases} 4 & \text{for } 0 < q_1 \leq 80 \\ 3 & \text{for } 80 < q_2 \leq 160 \\ 2 & \text{for } 160 < q_3 \end{cases}$$

Setup cost,  $C_1 = 250.0$

Holding cost,  $C_2 = .35 + .02 \times v_i(q_i)$

Percentage carrying cost,  $I = 2\%$

Shortage cost,  $C_3 = 2.0$

Lost sale cost,  $C_4 = 15.0$

Demand rate  $d(p) = ae^{-bp} = 1600e^{-0.9p}$

Deteriorating rate  $\alpha + \beta t_1 = 0.005 + 0.02t_1$

Using **Matlab**, we obtain the following tables.

Table 1 For  $B=1$

$V_i(q)$	$T^*$	$T_1^*$	$p^*$	$q^*$	$NP^*$
4	4	3.097	5.955	31.443	8.357 <sup>a</sup>
3	4	3.165	4.8981	80.5619	21.6454 <sup>a</sup>

2	4	3.2392	3.8374	209.7309	56.2257 <sup>a</sup>
4	5	3.813	6.2012	31.5473	6.6995 <sup>a</sup>
3	5	3.908	5.122	83.6026	17.6808 <sup>a</sup>
2	5	4.011	4.039	222.3343	46.8824 <sup>a</sup>
4	6	4.49	6.4617	30.4181	5.2993 <sup>a</sup>
3	6	4.6219	5.3602	82.3991	14.2802 <sup>a</sup>
2	6	4.7590	4.2510	224.8648	38.7516 <sup>a</sup>

Table 2 For B=.5

$V_i(q)$	$T^*$	$T_1^*$	$p^*$	$q^*$	$NP^*$
4	4	3.6575	6.1280	26.8953	7.1557 <sup>a</sup>
3	4	3.71066	5.0478	71.3908	18.9162 <sup>b</sup>
2	4	3.7586	3.9628	190.2113	50.2251 <sup>a</sup>
4	5	4.5400	6.4429	25.9027	5.3893 <sup>a</sup>
3	5	4.6143	5.3303	70.9102	14.6704 <sup>b</sup>
2	5	4.6820	4.2098	195.4772	40.2141 <sup>a</sup>
4	6	5.4047	6.7874	23.3671	3.9529 <sup>a</sup>
3	6	5.5038	5.6370	66.448	11.1310 <sup>b</sup>
2	6	5.5948	4.4750	190.6989	31.6751 <sup>a</sup>

Table 3 For B=0.1

$V_i(q)$	$T^*$	$T_1^*$	$p^*$	$q^*$	$NP^*$
4	4	3.7707	6.1632	26.2114	6.9324 <sup>a</sup>
3	4	3.8097	5.0753	70.0107	18.4546 <sup>b</sup>
2	4	3.8437	3.9835	187.5434	49.2989 <sup>a</sup>
4	5	4.6905	6.4938	24.9523	5.1481 <sup>a</sup>
3	5	4.7454	5.3693	69.0209	14.1642 <sup>b</sup>
2	5	4.7936	4.2385	191.8063	39.1887 <sup>a</sup>
4	6	5.5971	6.8583	22.2215	3.7084 <sup>a</sup>
3	6	5.6709	5.6907	64.0722	10.6059 <sup>b</sup>
2	6	5.7363	4.5137	186.0370	30.5911 <sup>a</sup>

<sup>a</sup> Feasible optimal solution

<sup>b</sup>  $q^*$  is too small for discount price, the solution is not feasible.

**Interpretation of Table 1, Table2 and Table 3**

1. The optimal average total net profit,  $NP^*$ , decreases as the backordering ratio,  $B$  decreases due to larger lost sale fraction.
2. The optimal lot size quantity  $q^*$  decreases with decreasing value of  $B$ .
3.  $NP^*$  increases as  $v_i(q)$  decreases
4.  $q^*$  increases as  $v_i(q)$  decreases
5. Optimal selling price  $p^*$  decreases with decreasing value of  $v_i(q)$
6. Optimal time with positive inventory  $T_1^*$  increasing with decreasing value of  $v_i(q)$ .

**7. Solution Procedure**

**Step1** Solve equations (3.5.9) and (3.5.10) simultaneously to find the values of  $p$  and  $T_1$  corresponding to a given value of  $T$  (say =4). We have

$$T_1 = 3.097 \text{ (say } T_1^*) \text{ and } p = 5.955 \text{ (say } p^*)$$

**Step 2** We find  $NP(T_1^*, p^*, T=4) = 8.357$  (3.7.1)

From Equation (3.5.7) and test for optimality of  $NP(T_1^*, p^*, T=4)$

We have

$$\frac{\partial^2}{\partial T_1^2} NP(T_1^*, p^*, T=4) = - 5.1307 < 0 \tag{3.7.2}$$

$$\frac{\partial^2}{\partial p^2} NP(T_1^*, p^*, T=4) = - 6.7757 < 0 \tag{3.7.3}$$

and

$$\frac{\partial^2}{\partial T_1 \partial p} NP(T_1^*, p^*, T=4) = - 0.0222 < 0 \tag{3.7.4}$$

We see that Hessian conditions (3.5.12) and (3.5.13) are satisfied. The value  $NP(T_1^*, p^*, T=4) = 8.357$  is thus maximum. Similarly other values given in table 1 and table 2 can be justified.

**Step 3** We obtain

$$NP(T_1^*, p^*, T=3) = 8.2613 \tag{3.7.5}$$

and

$$NP (T_1^*, p^*, T=5) = 5.4053 \tag{3.7.6}$$

obviously

$$NP (T_1^*, p^*, T=5) < NP (T_1^*, p^*, T=4) \tag{3.7.7}$$

and

$$NP (T_1^*, p^*, T=3) < NP (T_1^*, p^*, T=4) \tag{3.7.8}$$

Thus  $T^*=4$  and  $NP (T_1^*, p^*, T^*) = 8.357$

**Step 4** We obtain  $q^* = 31.0443$  from Equation (3.4.11)

### 8. Sensitivity Analysis

In this section, we use the same numerical example. The study aimed to analyze the influence of the parameters B, a, b,  $\alpha$ ,  $\beta$  with respect to  $T_1^*$ ,  $p^*$ ,  $q^*$  and  $NP^*$ . The sensitivity analysis results are presented in Table 4.

**Table 4**  $v_i(q) = 3 \quad T=4$

Parameters	Change in %	Actual value	$T_1^*$	$p^*$	$q^*$	NP
	-40%	0.30	3.7704	5.0644	70.5513	18.6364
<b>B</b>	-20%	0.40	3.7439	5.0571	70.9184	18.7597
	+20%	0.60	3.6673	5.0358	72.0272	19.1214
	+40%	0.70	3.6087	5.0196	72.9044	19.4024
Parameters	Change in %	Actual value	$T_1^*$	$p^*$	$q^*$	NP
	-40%	960	3.7107	5.0478	42.8380	11.3497
<b>a</b>	-20%	1280	3.7107	5.0478	57.1173	15.1329
	+20%	1920	3.7107	5.0478	85.6759	22.6994
	+40%	2240	3.7107	5.0478	99.9553	26.4827
Parameters	Change in %	Actual value	$T_1^*$	$p^*$	$q^*$	NP
	-40%	0.54	3.7107	5.7886	294.5434	130.071
<b>b</b>	-20%	0.72	3.7106	5.3256	145.0158	48.0279
	+20%	1.08	3.7106	4.8627	35.1475	7.7607
	+40%	1.26	3.7106	4.7304	17.3038	3.2750
Parameters	Change in %	Actual value	$T_1^*$	$p^*$	$q^*$	NP
	-40%	0.003	3.7133	5.0383	71.7687	19.0788
<b><math>\alpha</math></b>	-20%	0.004	3.7120	5.0431	71.5776	18.9974
	+20%	0.006	3.7093	5.0520	71.2466	18.8352
	+40%	0.007	3.7079	5.0570	71.0451	18.7544
Parameters	Change in %	Actual value	$T_1^*$	$p^*$	$q^*$	NP
	-40%	0.012	3.7363	4.9911	73.9293	19.9064
<b><math>\beta</math></b>	-20%	0.016	3.7236	5.0194	72.661	19.4069
	+20%	0.024	3.6974	5.0765	70.1242	18.4346
	+40%	0.028	3.6841	5.1053	68.8568	17.9623

Parameters	Change in %	Actual value	T1*	p*	q*	NP
	-40%	150	3.7106	5.0478	71.3908	18.9162
<b>C<sub>1</sub></b>	-20%	200	3.7106	5.0478	71.3912	18.9162
	+20%	300	3.7106	5.0478	71.3966	18.9162
	+40%	350	3.7106	5.0478	71.3966	18.9162
Parameters	Change in %	Actual value	T1*	p*	q*	NP
	-40%	1.2	3.6947	5.0434	71.5939	18.9913
<b>C<sub>3</sub></b>	-20%	1.6	3.7029	5.0457	71.4903	18.9527
	+20%	2.4	3.718	5.0499	71.301	18.8718
	+40%	2.8	3.7249	5.0518	71.2153	18.8488
Parameters	Change in %	Actual value	T1*	p*	q*	NP
	-40%	9	3.5248	4.996	73.6397	19.8107
<b>C<sub>4</sub></b>	-20%	12	3.6403	5.0284	72.252	19.2501
	+20%	18	3.7579	5.0609	70.8013	18.6946
	+40%	21	3.7919	5.0704	70.3613	18.5368

**Interpretation**

- NP\* increases with increasing value of B and a and decreases with increasing value of b, α, C<sub>3</sub> and C<sub>4</sub>. NP\* is lowly sensible to B, α, β, C<sub>3</sub> and C<sub>4</sub>; moderately sensible to a and highly sensible to b. It is insignificant sensible to C<sub>1</sub>.
- q\* increases with increasing value of a and decreases with increasing value of B, b, α, β, C<sub>3</sub> and C<sub>4</sub>. q\* is lowly sensible to B, α, β, C<sub>3</sub> and C<sub>4</sub> and highly sensible to a and b. It is insignificant sensible to C<sub>1</sub>.
- p\* decreases with increasing value of B and b and increases with increasing value of α, β, C<sub>3</sub> and C<sub>4</sub>. p\* is lowly sensible to B, b, α, β, C<sub>3</sub> and C<sub>4</sub>. It is insignificant sensible to a and C<sub>1</sub>.
- T<sub>1</sub>\* increases with increasing value of C<sub>3</sub> and C<sub>4</sub> and decreases with increasing value of B, α and β. T<sub>1</sub>\* is lowly sensible to B, α, β, C<sub>3</sub> and C<sub>4</sub>. It is insignificant sensible to a, b and C<sub>1</sub>.

**9. Conclusion**

In his study, a joint pricing and replenishment policy for a time dependent linear distribution deteriorating items. With partial backordering and quantity discounts has been developed. It is realistic to assume that the relations have some knowledge on the buyer's behavior such as their responses to shortages and price increase. There are

many situations where certain percentages of the buyers are willing to wait for services, depending on the comparative price of the product. The retailer should develop a replenishing and pricing policy which will ensure the largest net profit. It should be noted that a retailer can either shorten the replenishment cycle or increase the price to counteract the greater loss due to increasing lost sale fraction in order to maximize the profit. This can be anticipated since in the economic sense, a higher lost sale fraction means a more expensive shortage cost which when combined to other costs, would result in higher cost of the items. However higher price will result in decreasing demand and the shortage of the replenishment cycle will result in greater average demand and the shortening of the replenishment cycle will result in greater average replenishment cost. The situation is further complicated by the inclusion of quantity discounts. The conflicting influence is hard to predict without detailed analysis.

In this study, net profit was maximize by first solving simultaneously the optimal inventory holding time and the selling price. Due to the interactive influences, the selling price for given example remains almost constant as should be in good business practice. The inclusion of additional state variables in the model would provide better prediction and control although this would add to the complexity of the problem.

**References**

1. A.H. Kalro, M.M. Gohil, 1982. A lot size model with backlogging when the amount received is uncertain, *International Journal of Production Research* 20 (6) 775 - 786.
2. D. Rosenberg, 1979. A new analysis of a lot size model with partial backordering, *NavelResearch Logistics* 26 (2) 349- 353.
3. E.A. Elsayed, C. Teresi, 1983. Analysis of inventory systems with deteriorating items, *International Journal of Production Research* 21 449-460.
4. F. Rifaat, P.M. Wolfe, H.K. Eldin, 1991. An inventory model for deteriorating items, *Computers and Industrial Engineering* 20 89-94.
5. G.C. Philip, 1974. A generalized EOQ model for items with Weibull distribution deterioration, *AIIE Transaction* 6 159-162.
6. H.M. Wee, 1989. Optimal inventory policy with partial backordering, *Optimal Control Applications and Methods* 10 181-187.
7. H.M. Wee, 1993. Economic production lot size for deteriorating items with partial backordering, *Computers and Industrial Engineering* 24 449-458.
8. H.M. Wee, 1995. Joint pricing and replenishment policy for deteriorating inventory with declining market, *International Journal of Production Economics* 40 163-171.
9. Hui-Ming Wee 1999. Deteriorating inventory model with quantity discount, pricing and partial backordering .Department of Industrial Engineering, College of Engineering, Chung Yuan Christian University, Chungli 32023, Taiwan *Int. J. Production Economics* 59 511-518
10. J. Liebman, L. Lasdon, L. Schrage A. 1986. *Waren, Modeling and Optimization with GINO*, The Scientific Press, South San Francisco.
11. M.A. Cohen, 1977. Joint pricing and ordering policy for exponentially decaying inventory with known demand, *Naval Research Logistic Quarterly* 24 257-268.
12. M.R. Spiegel, 1960. *Applied Differential Equations*, Prentice- Hall, New York,
13. P.M. Ghare, G.F. Schrader, 1963. A model for exponentially decaying inventory, *Journal of Industrial Engineering* 14 238-243.
14. P.L. Abad, 1988. Joint price and lot-size determination when supplier offers incremental quantity discounts, *Journal of Operational ResearchSociety* 39 603 - 607.
15. R.P. Covert, G.C. Philip, 1973. An EOQ model for items with Weibull distribution deterioration, *AIIE Transactions* 5 323- 326.
16. R.E. Wapole, R.H. Myers, 1978. *Probability and Statistic for Engineers and Scientists*, Macmillan Publishing Co., New York,
17. S.P. Aggarwal, 1979. Note on: an order level for deteriorating items by Y, K. Shah, *AIIE Transactions* 11 344- 346.
18. S. Eilon, R.V. Mallaya, 1966. Issuing and pricing policy of semiperishables, *Proceedings of 4th International conference on Operational research*, Wiley-Inter science, New York,.
19. S. Kang, I. Kim, 1983. A study on the price and production level of the deteriorating inventory system, *International Journal of Production Research* 21 449-460.
20. S.P. Aggarwal, C.K. Jaggi, 1989. Ordering policy for decaying inventory, *International Journal of Systems Science* 20 151-155.
21. S. Love, 1979. *Inventory Control*, McGraw-Hill, New York,
22. U. Dave, 1979. On a discrete-in-time order-level inventory model for deteriorating items, *Operational Research Quarterly* 30 349- 354.
23. Y.K. Shah, 1976. An order-level lot-size inventory model for deteriorating items, *AIIE Transactions* 9 108-112.
24. Y.C. Shiu, 1990. An inventory model for perishable items in a lot-size system with quantity discounts, *European Journal of Operational Research* 45 260 - 264.