

Forbidden induced subgraph of the Comparability Graph and Three Colored Posets

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ABSTRACT

The cover-incomparability graph of a poset P is the edge-union of the covering and the incomparability graph of P . As a continuation of the study of 3-colored diagrams we characterize some forbidden \triangleleft -preserving subposets of the posets whose cover-incomparability graph contains one of the forbidden induced subgraph of the comparability graph.

1. Introduction

Cover-incomparability graphs of posets, or shortly C-I graphs, were introduced in [2] as the underlying graphs of the standard interval function or transit function on posets (for more on transit functions in discrete structures [3, 4, 5, 6, 11]). On the other hand, C-I graphs can be defined as the edge-union of the covering and incomparability graph of a poset; in fact, they present the only non-trivial way to obtain an associated graph as unions and/or intersections of the edge sets of the three standard associated graphs (i.e. covering, comparability and incomparability graph). In the paper that followed [9], it was shown that the complexity of recognizing whether a given graph is the C-I graph of some poset is in general NP-complete. In [1] the problem was investigated for the classes of split graphs and block graphs, and the C-I graphs within these two classes of graphs were characterized. This resulted in a linear-time recognition algorithms for C-I block and C-I split graphs. It was also shown in [1] that whenever a C-I graph is a chordal graph, it is necessarily an interval graph, however a structural characterization of C-I interval graphs (and thus C-I chordal graphs) is still open. C-I distance-hereditary graphs have been characterized and shown to be efficiently recognizable [10]. Let $P = (V; \leq)$ be a poset. If $u \leq v$ but $u \neq v$, then we write $u < v$. For $u, v \in V$ we say that v covers u in P if $u < v$ and there is no w in V with $u < w < v$. If $u \leq v$ we will sometimes say that u is below v , and that v is above u . Also, we will write $u \triangleleft v$ if v covers u ; and $u \triangleleft\triangleleft v$ if u is below v but not covered by v . By $u \parallel v$ we denote that u and v are incomparable. Let V' be a nonempty subset of V . Then there is a natural poset $Q = (V'; \leq')$, where $u \leq' v$ if and only if $u \leq v$ for any $u, v \in V'$. The poset Q is called a subposet of P and its notation is simplified to $Q = (V'; \leq)$. If, in addition, together with any two comparable elements u and v of Q , a chain of shortest length between u and v of P is also in Q , we say that Q is an isometric subposet of P . Recall that a poset P is dual to a poset Q if for any $x, y \in P$ the following holds: $x \leq y$ in P if and only if $y \leq x$ in Q . Given a poset P , its cover-incomparability graph G_P has V as its vertex set, and uv is an edge of G_P if $u \triangleleft v$, $v \triangleleft u$, or u and v are incomparable. A graph that is a cover-

incomparability graph of some poset P will be called a C-I graph.

Lemma 1 [2] Let P be a poset and G_P its C-I graph. Then

- (i) G_P is connected;
- (ii) vertices in an independent set of G_P lie on a common chain of P ;
- (iii) an antichain of P corresponds to a complete subgraph in G_P ;
- (iv) G_P contains no induced cycles of length greater than 4.

2. Three-colored diagrams

A 3-coloured diagram Q in [13] is explained as follows. Let G be a C-I graph and H be an induced subgraph of G . We note that there can be different \triangleleft -preserving subposets Q_i of some posets with GQ_i isomorphic to the subgraph H . Let u, v, w be an induced path in the direction from u to v in H . There are four possibilities in which u, v and w can be related in the \triangleleft -preserving subposets. It is possible to have $u \triangleleft v$, $u \parallel v$, $v \triangleleft w$ and $v \parallel w$. Each case will appear as a \triangleleft -preserving subposet of four different posets. If $u \triangleleft v$ and $v \triangleleft w$ in a subposet, then $u \triangleleft v \triangleleft w$ is a chain in the subposet and u, v, w is an induced path in H . If there is either $u \parallel v$ or $v \parallel w$ in a subposet Q , then there should be another chain from u to w in Q in order to have u, v, w an induced path in H . We try to capture this situation using the idea of 3-colored diagram. Suppose in \triangleleft -preserving subposet Q of a poset P , there exists two elements u, v which is always connected by some chain of length three in Q . Let w be an element in Q such that either both uw and vw are red edges or any one of them is a red edge. Then in order to have a chain between u and v , there must exist an element x in Q so that u, x, v form a chain in Q . When both edges are normal, then we have the chain u, w, v in Q and hence the chain u, x, v is not required in this case. We denote the chain u, x, v by dashed lines between ux and xv in order to specify that it is possible to have the presence or absence of the chain u, x, v in Q . The presence of the chain u, x, v implies that either both of the edges uw and wv are red edges or one of them is a red edge. The absence of the chain implies that both uw and wv are

normal edges in Q. We call posets having the above mentioned diagrams as 3-colored diagrams.

Theorem 2: (Theorem 1,[8]): Let G be a class of graphs with a forbidden induced subgraphs characterization. Let $\mathcal{F} = \{P \mid P$ is

a poset with $G_{TP} \in G\}$. Then \mathcal{F} has a characterization by forbidden \triangleleft -preserving subposets.

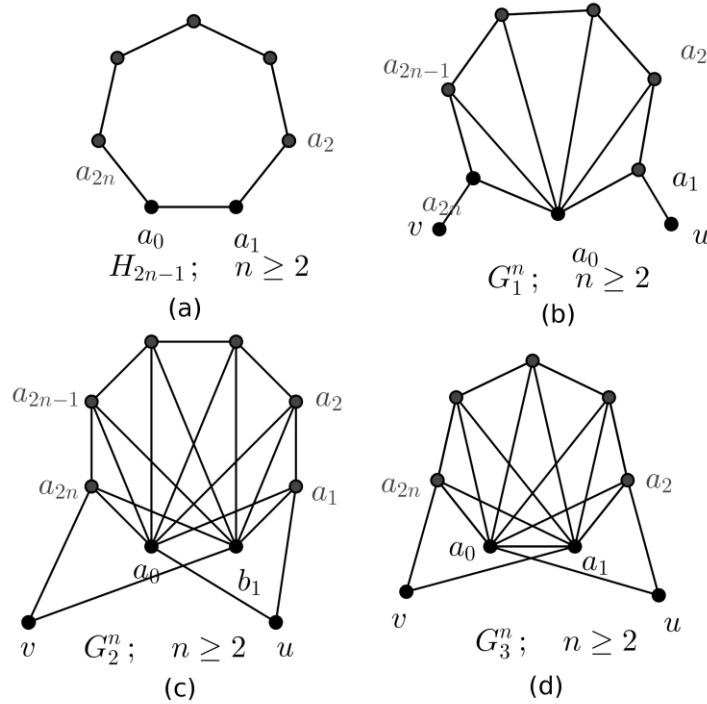


FIGURE 1: Families of forbidden induced subgraphs for comparability graph [14]

Let $\mathcal{F}(G)$ be a collection of forbidden induced subgraphs. We consider the posets whose cover-incomparability graph G_P contains the graphs in $\mathcal{F}(G)$.

3. \triangleleft -preserving subposets of posets whose C-I graphs contains the graphs in $\mathcal{F}(G)$

We have the following theorem regarding graph family $\mathcal{F}(G)$

Theorem 3: If P is a poset, then G_P contains $\mathcal{F}(G_2^n)$, $n \geq 2$ if and only if P contains the 3-colored diagrams Q_2 from Figure 2 and their duals.

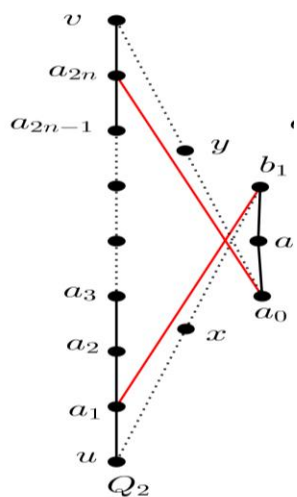


Figure 2 :Forbidden 3-colored diagrams for posets whose C-I graphs contains G_2^n depicted in Figure 1 (c).

Proof. Suppose P contains 3-colored diagrams Q_2 . Then since $a_2, a_3, \dots, a_{2n-1}$ are incomparable with a_0 and b_1 in P, the set of vertices $\{u, a_0, a_1, b_1, a_2, a_3, \dots, a_{2n-1}, a_{2n}, v\}$ induce the graph G_2^n , $n \geq 2$ from Figure 1 (c).

Conversely, suppose $G_P \in \mathcal{F}(G)$. Then G_P contains an induced subgraph G_2^n , $n \geq 2$ shown in Figure 1(c), with vertices labeled by $u, a_0, a_1, b_1, a_2, a_3, \dots, a_{2n-1}, a_{2n}, v$. The set of vertices $\{u, a_2, a_4, \dots, a_{2n}\}$ is an independent set in G_2^n , $n \geq 2$. Therefore these vertices lie on a common chain in P (by Lemma 1(ii)) and they are not in a covering relation. Denote the chain containing $\{u, a_2, a_4, \dots, a_{2n}\}$ by C. Then the following cases (i) and (ii) cannot occur.

(i): $a_{2i-4} \triangleleft a_{2i} \triangleleft a_{2i-2}$, $i \geq 3$:

Since a_{2i-3} and a_{2i} are nonadjacent in G_2^n , they lie on a common chain in P.

$a_{2i-3} \triangleleft a_{2i}$: then $a_{2i-3} \triangleleft a_{2i} \triangleleft a_{2i-2}$ in P, contradicting a_{2i-3} and a_{2i-2} are adjacent in G_2^n .

$a_{2i} \triangleleft a_{2i-3}$: then $a_{2i-4} \triangleleft a_{2i} \triangleleft a_{2i-3}$ in P, contradicting a_{2i-4} and a_{2i-3} are adjacent in G_2^n .

The same contradiction arise if $a_{2i-2} \triangleleft a_{2i} \triangleleft a_{2i-4}$.

(ii): $a_{2i-2} \triangleleft a_{2i-4} \triangleleft a_{2i}$, $i \geq 3$:

Since a_{2i-4} and a_{2i-1} are nonadjacent in G, they lie on a common chain in P.

$a_{2i-4} \triangleleft a_{2i-1}$: then $a_{2i-2} \triangleleft a_{2i-4} \triangleleft a_{2i-1}$ in P, contradicting a_{2i-2} and a_{2i-1} are adjacent in G_2^n .

$a_{2i-1} \triangleleft a_{2i-4}$: then $a_{2i-1} \triangleleft a_{2i-4} \triangleleft a_{2i}$ in P , contradicting a_{2i-1} and a_{2i} are adjacent in G_2^n .

The same contradiction arise if $a_{2i} \triangleleft a_{2i-4} \triangleleft a_{2i-2}$, $i \geq 3$.

The only possible cases are $a_{2i-4} \triangleleft a_{2i-2} \triangleleft a_{2i}$, and $a_{2i} \triangleleft a_{2i-2} \triangleleft a_{2i-4}$. Without loss of generality, assume that $a_{2i-4} \triangleleft a_{2i-2} \triangleleft a_{2i}$. Since a_{2i-3} is adjacent to a_{2i-4} and a_{2i-2} and a_{2i-1} is adjacent to a_{2i-2} and a_{2i} in G_2^n , we have $a_2 \triangleleft a_3 \triangleleft a_4 \triangleleft \dots \triangleleft a_{2n}$. As a_1 is adjacent to u and a_2 , we have $u \triangleleft a_1 \triangleleft a_2$. Since v is adjacent to a_{2n} and nonadjacent to all other vertices of C implies $a_{2n} \triangleleft v$ and hence $u \triangleleft a_1 \triangleleft a_2 \triangleleft a_3 \triangleleft a_4 \triangleleft \dots \triangleleft a_{2n} \triangleleft v$. The vertices a_0 and b_1 are nonadjacent in G_2^n these vertices lie on a common chain in P . The chain C does not

contain a_0 and b_1 , since a_0 is adjacent to all vertices of C in G_2^n other than v and b_1 is adjacent to all other vertices of C in G_2^n other than u . Also $a_0 \triangleleft u$, since a_0 and a_1 are adjacent and $v \triangleleft b_1$, since b_1 and a_{2n} are adjacent in G_2^n . Now $b_1 \triangleleft a_0$, since b_1 adjacent to both v and a_{2n} in G_2^n . Thus there exists an element a such that $a_0 \triangleleft a \triangleleft b_1$; then we have two possibilities.

Case (1): $a_1 \triangleleft b_1$ in P : again we have two possibilities.

Subcase (1.1) $a_0 \triangleleft a_{2n}$ in P : take (a_1, b_1) and (a_0, a_{2n}) as normal edges and avoid all dashed lines in Figure 2 to obtain the \triangleleft -preserving subposet P_4 in Figure 3.

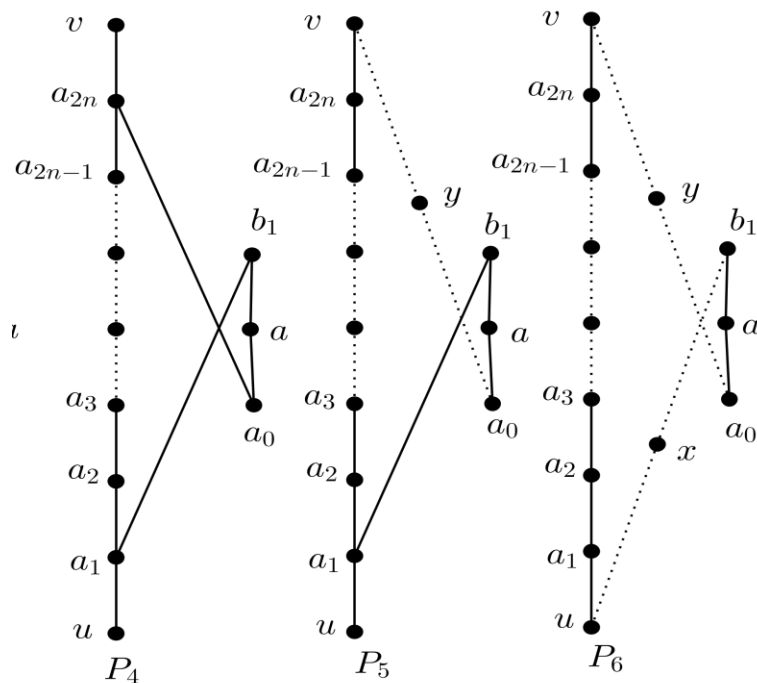


Figure 3: \triangleleft -preserving subposets corresponding to Q_2 .

Subcase (1.2) $a_{2n} \parallel a_0$ in P : take the chain from v to a_0 through y to obtain the \triangleleft -preserving subposet P_5 in Figure 3.

Case (2): $a_1 \parallel b_1$ in P : take the chain from u to b_1 through x to obtain the \triangleleft -preserving subposet P_6 in Figure 3.

All posets in Figure 3 are represented by a single 3-colored diagram Q_2 in Figure 2. \square

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