

# On Pairwise Basis for Bitopological Spaces

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## ABSTRACT

To investigate and establish the notion of pairwise basis for bitopological spaces is the primary focus of this research manuscript. Furthermore, this notion of pairwise basis for bitopological spaces is to study and establish generalization of selective results and theorems, analogues to well-known theorems of classical topology, for bitopological spaces.

## 1. Introduction and Preliminaries

Kelly [1] was the first to introduce the concept of bitopological spaces in 1963. This new concept of bitopological spaces is used to study non-symmetric functions that introduce two arbitrary topologies on  $X$ . In the same research work, J. C. Kelly generalizes the concept of selective separation properties of topological space to bitopological spaces in the form of pairwise Hausdorff, pairwise regular and pairwise normal. Further work in the field of bitopological spaces is carried out by Kim [2], Fletcher [3] et al., Patty [4], Pervin [5], Saegrove [6] and many others. Our emphasis in present research document is to develop idea of pairwise basis for a bitopological space and then with the support this concept of pairwise basis, well known theorems of classical topology will be generalized to bitopological spaces.

If  $\tau_1, \tau_2$  are any two topologies on a non-empty set  $X$  then,  $(X, \tau_1, \tau_2)$  is named as a bitopological [1] space on  $X$ . For any subset  $A$  of  $(X, \tau_1, \tau_2)$ ,  $\tau_1\text{-cl}(A)$  and  $\tau_2\text{-cl}(A)$  denote closure of  $A$  with respect to  $\tau_1$  and  $\tau_2$  respectively. Further,  $\tau_1\text{-open}$  ( $\tau_1\text{-closed}$ ) and  $\tau_2\text{-open}$  ( $\tau_2\text{-closed}$ ) will be used to denote open (closed) set in a bitopological space  $(X, \tau_1, \tau_2)$  with respect to  $\tau_1$  and  $\tau_2$  respectively. Any subset of a bitopological space  $(X, \tau_1, \tau_2)$  will be named as pairwise open (closed) if and only if subset under consideration is open (closed) with respect to  $\tau_1$  and  $\tau_2$ .

## 2. Pairwise Basis for a Bitopological Space

For a bitopological space, a weaker form of basis, named as pairwise basis, will be defined as under:

**Definition ( Pairwise Basis) 1.** Let  $(X, \tau_1, \tau_2)$  be a bitopological space, then a collection  $\mathbf{b}_p \subseteq P(X)$  is said to be pairwise basis of the bitopological space if and only if

- $\mathbf{b}_p \subseteq \tau_1 \cap \tau_2$ , i. e.,  $\mathbf{b}_p$  is a collection of pairwise open sets and
- for each  $x$  in  $X$  and for each pairwise open set  $U$  containing  $x$ , there exists  $B$  in  $\mathbf{b}_p$  such that  $x \in B \subseteq U$ .

**Theorem 1.** For a bitopological space  $(X, \tau_1, \tau_2)$ ,  $\mathbf{b}_p \subseteq \tau_1 \cap \tau_2$  is pairwise basis for  $(X, \tau_1, \tau_2)$  if and only if every pairwise open set can be expressed as union of members of  $\mathbf{b}_p$ .

**Proof.** ( $\Rightarrow$ ) Suppose that  $\mathbf{b}_p \subseteq \tau_1 \cap \tau_2$  is pairwise basis for  $(X, \tau_1, \tau_2)$  and  $U$  is any pairwise open set in the bitopological space. For any  $x$  in  $U$ , there exists  $B_x \in \mathbf{b}_p$  such that  $x \in B_x \subseteq U$ , i.e.,  $\{x\} \subseteq B_x \subseteq U$ . Taking union for the variation of  $x$  over  $U$ , we get

$$\bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B_x \subseteq \bigcup_{x \in U} U.$$

Thus

$$U = \bigcup_{x \in U} B_x.$$

( $\Leftarrow$ ) Suppose that every pairwise open set can be expressed as union of members of  $\mathbf{b}_p$ . To show that  $\mathbf{b}_p$  is pairwise basis of  $(X, \tau_1, \tau_2)$ . First condition is trivial. Further, we select arbitrary  $x \in X$  and also any set  $U$ , which is pairwise open set such that  $x \in U$ . Since  $U$  is pairwise open therefore,

$$U = \bigcup_{\alpha \in \Lambda} B_\alpha,$$

As  $x \in U$ , so it is possible to find at least one member in above union such that  $x \in B_\alpha \subseteq U$ . Hence  $\mathbf{b}_p$  is pairwise basis of  $(X, \tau_1, \tau_2)$ .

**Theorem 2.** If  $\mathbf{b}_p$  is a collection of subsets of a non-empty set  $X$  such that

$$X = \bigcup B,$$

where  $B \in \mathbf{b}_p$ , then  $\mathbf{b}_p$  forms pairwise basis for some bitopological space  $(X, \tau_1, \tau_2)$  if and only if for each  $B_1, B_2 \in \mathbf{b}_p$  and for each  $x \in B_1 \cap B_2$  it is possible to find some  $B$  in  $\mathbf{b}_p$  satisfying  $x \in B \subseteq B_1 \cap B_2$ .

**Proof.**

( $\Rightarrow$ ) Assume that  $\mathbf{b}_p$  forms pairwise basis for some bitopological space  $(X, \tau_1, \tau_2)$ . Consider any two members  $B_1, B_2$  of  $\mathbf{b}_p$  such that  $x \in B_1 \cap B_2$  is arbitrary. Since intersection of pairwise open sets is also pairwise open therefore, there exists  $B \in \mathbf{b}_p$  such that  $x \in B \subseteq B_1 \cap B_2$ .

( $\Leftarrow$ ) Suppose the given condition holds. To establish that  $\mathbf{b}_p$  is pairwise basis for some bitopological space  $(X, \tau_1, \tau_2)$ . Consider  $\tau_1$  and  $\tau_2$  as follows:

$\tau_1 = \{B_1 \subseteq X: B_1 \text{ is union of members of } \mathbf{b}_p\} \cup \{\emptyset\}$   
 $\tau_2 = \{B_2 \subseteq X: B_2 \text{ is union of members of } \mathbf{b}_p\} \cup \{\emptyset\}.$

To prove the desired result, it is sufficient to show that  $\tau_1$  and  $\tau_2$  are arbitrary topologies on  $X$ . First, consider the case of  $\tau_1$ . Clearly,  $\emptyset \in \tau_1$  and  $X$ , being union of members of  $\mathbf{b}_p$  also belongs to  $\tau_1$ . Let  $G$  and  $H$  are two arbitrary members of  $\tau_1$ . Construction of  $\tau_1$  suggests that

$$G = \bigcup B_a$$

where  $B_a$  is member of  $\mathbf{b}_p$  and

$$H = \bigcup B_b$$

where  $B_b$  is member of  $\mathbf{b}_p$

Let  $x \in G \cap H$  be arbitrary. This means  $x \in B_a \subseteq G$  and  $x \in B_b \subseteq H$ , where  $B_a$  and  $B_b$  are members of  $\mathbf{b}_p$ . Therefore,  $x \in B_a \cap B_b \subseteq G \cap H$ . Using given condition, there exists  $B \in \mathbf{b}_p$  such that  $x \in B \subseteq B_a \cap B_b \subseteq G \cap H$ , i.e.,  $\{x\} \in B \subseteq G \cap H$  this implies

$$\bigcup_{x \in G \cap H} \{x\} \subseteq \bigcup_{x \in G \cap H} B_x \subseteq \bigcup_{x \in G \cap H} G \cap H.$$

From this expression one can easily conclude that

$$G \cap H = \bigcup_{x \in G \cap H} B_x$$

i.e.,  $G \cap H$  can be expressed as union of members of  $\mathbf{b}_p$ . Thus,  $G \cap H \in \tau_1$ .

Again consider  $\{A_a: a \in \Lambda\}$  of members of  $\tau_1$ , which is an arbitrary collection. As each  $A_a$  is union of members of  $\mathbf{b}_p$  therefore,

$$\bigcup_{a \in \Lambda} A_a$$

is also union of members of  $\mathbf{b}_p$ . Hence,  $\tau_1$  is a topology on  $X$  and every member of  $\tau_1$  can be expressed as union of members of  $\mathbf{b}_p$ . Proceeding on the same lines, it can be proved that  $\tau_2$  is a topology on  $X$  and every member of  $\tau_2$  can be expressed as union of members of  $\mathbf{b}_p$ . Combining these facts one can conclude that  $\mathbf{b}_p$  constitutes pairwise basis for the bitopological space  $(X, \tau_1, \tau_2)$ .

**Theorem 3.** Necessary condition for a collection  $\mathbf{b}_p$  of subsets of a non-empty set  $X$  to form pairwise basis for a bitopological space  $(X, \tau_1, \tau_2)$  is that

$$X = \bigcup B,$$

where  $B \in \mathbf{b}_p$ .

**Proof.** Since  $X$  is a pairwise open set therefore, for any member  $x$  of  $X$  it is possible to find  $B \in \mathbf{b}_p$  such that  $x \in B_x \subseteq X$ , i.e.,  $\{x\} \subseteq B_x \subseteq X$  this implies that

$$\bigcup_{x \in X} \{x\} \subseteq \bigcup_{x \in X} B_x \subseteq \bigcup_{x \in X} X.$$

From this expression one can easily conclude that

$$X = \bigcup_{x \in X} B_x.$$

**Remark.** Above stated condition is not sufficient condition.

### 3. Conclusion

With the help of new idea of pairwise basis for bitopological space, some well-known theorems/ results of classical topological space are generalized to bitopological spaces. But major drawback of this concept of pairwise basis is that the two arbitrary topologies of the bitopological space reduce to two equivalent topologies.

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