RESEARCH REVIEW JOURNALS
Volume-03
Issue-08
August-2018

ISSN: 2455-3085 (Online)
RESEARCH REVIEW International Journal of Multidisciplinary
www.rrjournals.com [UGC Listed Journal]

Common Fixed Point Theorem for Two Pairs of Set Valued Mappings in G - Metric Space

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ARTICLE DETAILS

Article History
Published Online: 07 August 2018

Keywords
Fixed point, complete metric space, set-valued mapping, G – metric space.

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ABSTRACT

In this paper, we prove fixed point theorem and common fixed point theorem for two pairs of set valued mappings in $G-$ metric spaces. Further, the famous Banach's contraction principle and some of its generalizations and variants are realizable as special cases of our results.

1. Introduction

There are a lot of fixed and common fixed point results in different types of spaces. Metric fixed point theory is an important mathematical discipline because of its applications in areas such as variational and linear inequalities, optimization, and approximation theory. Generalizations of metric spaces were proposed by Gahler [5, 6] (called 2-metric spaces) and Dhage [3,4]. Hsiao [7] showed that, for every contractive definition, $x_n = T^n x_0$, every orbit is linearly dependent, thus rendering fixed point theorems in such spaces trivial. Unfortunately, it was shown that certain theorems involving Dhage's D-metric spaces are flawed, and most of the results claimed by Dhage and others are invalid. These errors were pointed out by Mustafa and Sims in [9], among others. They also introduced a valid generalized metric space structure, which they call G-metric spaces. Some other papers dealing with G-metric spaces are those in [2, 8–12].

2. Preliminaries

Definition 2.1 [9]. Let X be a nonempty set, and let $G: X \times X \times X \to \mathbb{R}^+$ be a function satisfy the following axioms:
(G1) $G(x,y,z) = 0$, iff $x = y = z$,
(G2) $0 < G(x,x,y)$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x,x,y) \leq G(x,y,z)$, for all $x,y,z \in X$ with $z \neq y$,
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$ (symmetry in all three variables),
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$ (rectangle inequality)
Then the function G is called a generalized metric, or, more specifically, a $G-$ metric on X, and the pair (X,G) is called a $G-$metric space.

Definition 2.2 [9]. Let (X,G) and (X',G') be $G-$metric spaces and let $f:(X,G) \longrightarrow (X',G')$ be function, then f is said to be $G-$continuous at a point $a \in X$; if given $\epsilon > 0$, there exists $\delta > 0$ such that $x,y \in X; G(a,x,y) < \delta$ implies that

$$G'(f(a),f(x),f(y)) < \epsilon.$$

A function f is $G-$ continuous on X if and only if it is $G-$ continuous at all $a \in X$.

Proposition 2.3 [9]. Let (X,G) and (X',G') be $G-$metric spaces, then a function $f: X \longrightarrow X'$ is $G-$continuous at a point $x \in X$ if and only if it is $G-$sequentially continuous at x; that is, whenever $\{x_n\}$ is $G-$ convergent to x, $\{f(x_n)\}$ is $G-$ convergent to $f(x)$.

Definition 2.4 [9]. Let (X,G) be a $G-$metric space, let $\{x_n\}$ be sequence of points of X; therefore, we say that $\{x_n\}$ is $G-$ convergent to x if

$$\lim_{n,m\to\infty} G(x,x_n,x_m) = 0;$$

that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \epsilon$ for all $n,m \geq N$. We call x, the limit of the sequence and write $x_n \longrightarrow x$ or $lim\, x_n = x$.

Proposition 2.5 [9]. Let (X,G) be a $G-$metric space. Then the following are equivalent:
(1) $\{x_n\}$ is $G-$convergent to x
(2) $G(x_n,x,x) \longrightarrow 0$, as $n \longrightarrow \infty$
(3) $G(x_n,x_n,x) \longrightarrow 0$, as $n \longrightarrow \infty$
(4) $G(x_m,x_n,x) \longrightarrow 0$, as $n \longrightarrow \infty$

Definition 2.6 [9]. Let (X,G) be a $G-$metric space. A sequence $\{x_n\}$ is called a $G-$ Cauchy if, for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(x_m,x_n,x_l) < \epsilon$, for all $n,m,l \geq N$; that is, $G(x_m,x_n,x_l) \longrightarrow 0$, as $n,m,l \longrightarrow \infty$.

Proposition 2.7 [9]. Let (X,G) be a $G-$metric space. Then the following are equivalent:
(1) The sequence $\{x_n\}$ is $G-$Cauchy.
(2) For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \epsilon$ for all $n,m \geq N$.

Proposition 2.8 [9]. Let (X,G) be a $G-$metric space. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables.

Definition 2.9 [9]. A $G-$metric space (X,G) is called a symmetric $G-$metric space if $G(x,y,y)=G(y,x,x)$, for all $x,y\in X$.

Proposition 2.10 [9]. Every $G-$metric space (X,G) defines a metric space (X,d_G) by $d_G(x,y)=G(x,y,y)=G(y,x,x)$, for all $x,y\in X$. Note that, if (X,G) is a symmetric $G-$metric space, then $d_G(x,y)=2\,G(x,y,y)$, for all $x,y\in X$. However, if (X,G) is not symmetric, then it holds by the $G-$metric properties that $\frac{3}{2}\,G(x,y,y)\le d_G(x,y)\le 3\,G(x,y,y)$, for all $x,y\in X$.

In general, these inequalities cannot be improved.

Proposition 2.11 [9]. A $G-$metric space (X,G) is $G-$complete if every Cauchy sequence in (X,G) is $G-$convergent.

Proposition 2.12 [9]. Let (X,G) be a $G-$metric space. Then, for any $x,y,z,a\in X$ it follows that

(1) if $G(x,y,z)=0$, then $x=y=z$,

(2) $G(x,y,z)\le G(x,x,y)+G(x,x,z)$,

(3) $G(x,y,y)\le 2G(y,x,x)$,

(4) $G(x,y,z)\le G(x,a,z)+G(a,y,z)$,

(5) $G(x,y,z)\le\frac{2}{3}[G(x,a,a)+G(y,a,a)+G(z,a,a)]$.

Example 2.13 [9]. Let $(\mathbb{R},d)$ be the usual metric space. Define G_a by $G_a(x,y,z)=d(x,y)+d(y,z)+d(x,z)$ for all $x,y,z\in\mathbb{R}$. Then it is clear that $(\mathbb{R},G_a)$ is a $G-$metric space.

Example 2.14 [9]. Let $X=\{a,b\}$. Define G on $X\,\mathrm{x}\,X\,\mathrm{x}\,X$ by $G(a,a,a)=G(b,b,b)=0$, $G(a,a,b)=1, G(a,b,b)=2$ and extend G to $X\,\mathrm{x}\,X\,\mathrm{x}\,X$ by using symmetry in the variables. Then it is clear that (X,G) is $G-$metric space.

3. Main Results

Theorem 3.1. Let (X,G_1) and (Y,G_2) be symmetric and complete G-metric spaces. Let T_1,T_2 and T_3 be mappings of X into $B(Y)$ and S_1,S_2 and S_3 be mappings of Y into $B(X)$ satisfying the inequalities

$$G_1\left(S_1T_1x,\ S_2T_2x',\ S_3T_3x''\right)$$

$$=\ C\max\left\{\begin{matrix}G_1\left(x,x',x''\right),G_1\left(x,S_1T_1x,x''\right),G_1\left(x',S_2T_2x',x\right),\\ G_1\left(x'',S_3T_3x'',x'\right),G_2\left(T_1x,T_2x',T_3x''\right)\end{matrix}\right\}\qquad(1)$$

$$G_2\left(T_2S_1y,\ T_3S_2y',\ T_1S_3y''\right)$$

$$=C\max\left\{\begin{matrix}G_2\left(y,y',y''\right),G_2\left(y,T_2S_1y,y''\right),G_2\left(y',T_3S_2y',y\right),\\ G_2\left(y'',T_1S_3y'',y'\right),G_1\left(S_1y,S_2y',S_3y''\right)\end{matrix}\right\}\qquad(2)$$

For all $x,x',x''\in X$ and $y,y',y''\in Y$ and $0\le C<1$. If one of the mapping T_1,T_2,T_3,S_1,S_2 and S_3 is continuous, then S_1T_1, S_2T_2 and S_3T_3 have a unique common fixed point $z\in X$ and T_2S_1, T_3S_2 and T_1S_3 have a common fixed point $w\in$ Y. Further $T_1z=T_2z=T_3z=\{w\}$ and $S_1w=S_2w=S_3w=\{z\}$.

Proof:

Let x be an arbitrary point in X, and define the points

$y_1\in T_1x,\ \ x_2\in S_1y_1,\ \ y_2\in T_2x_2,\ \ x_3\in S_2y_2,\ \ y_3\in T_3x_3,\ \ x_4\in S_3y_3.$

Define a sequence $\{x_n\}$ and $\{y_n\}$ in $B(X)$ and $B(Y)$ respectively, by choosing a point,

$$\begin{aligned}y_{3n-2}&\in T_1x_{3n-2}=Y_{3n-2}\\ x_{3n-1}&\in S_1y_{3n-2}=X_{3n-1}\\ y_{3n-1}&\in T_2x_{3n-1}=Y_{3n-1}\\ x_{3n}&\in S_2y_{3n-1}=X_{3n}\\ y_{3n}&\in T_3x_{3n}=Y_{3n}\\ x_{3n+1}&\in S_3y_{3n}=X_{3n+1}\quad\text{for each }n\in\mathbb{N}\end{aligned}$$

Using (1) and (2), we have

$$\begin{aligned}G_1(X_{3n+1},X_{3n},X_{3n-1})&=G_1(S_3T_3X_{3n},S_2T_2X_{3n-1},S_1T_1X_{3n-2})\\&=G_1(S_1T_1X_{3n-2},S_2T_2X_{3n-1},S_3T_3X_{3n})\\&\le C\max\left\{\begin{matrix}G_1(X_{3n-2},X_{3n-1},X_{3n}),\\G_1(X_{3n-2},X_{3n-1},X_{3n}),\\G_1(X_{3n-1},X_{3n},X_{3n-2}),\\G_1(X_{3n},X_{3n+1},X_{3n-1}),\\G_2(T_1X_{3n-2},T_2X_{3n-1},T_3X_{3n})\end{matrix}\right\}\\&\le C\max\left\{\begin{matrix}G_1(X_{3n-2},X_{3n-1},X_{3n}),\\G_1(X_{3n-2},X_{3n-1},X_{3n}),\\G_1(X_{3n-2},X_{3n-1},X_{3n}),\\G_1(X_{3n+1},X_{3n},X_{3n-1}),\\G_2(Y_{3n-2},Y_{3n-1},Y_{3n})\end{matrix}\right\}\end{aligned}$$

$$G_1(X_{3n+1},X_{3n},X_{3n-1})\le C\max\left\{\begin{matrix}G_1(X_{3n-2},X_{3n-1},X_{3n}),\\ \\G_2(Y_{3n-2},Y_{3n-1},Y_{3n})\end{matrix}\right\}$$

$$\begin{aligned}G_2(Y_{3n+1},Y_{3n},Y_{3n-2})&=G_2(T_1S_3Y_{3n},T_3S_2Y_{3n-1},T_2S_1Y_{3n-2})\\&=G_2(T_2S_1Y_{3n-2},T_3S_2Y_{3n-1},T_1S_3Y_{3n})\\&\le C\max\left\{\begin{matrix}G_1(Y_{3n-2},Y_{3n-1},Y_{3n}),\\G_1(Y_{3n-2},Y_{3n-1},Y_{3n}),\\G_1(Y_{3n-1},Y_{3n},Y_{3n-2}),\\G_1(Y_{3n},Y_{3n+1},Y_{3n-1}),\\G_2(S_1Y_{3n-2},S_2Y_{3n-1},S_3Y_{3n})\end{matrix}\right\}\\&\le C\max\left\{\begin{matrix}G_1(Y_{3n-2},Y_{3n-1},Y_{3n}),\\G_1(Y_{3n-2},Y_{3n-1},Y_{3n}),\\G_1(Y_{3n-2},Y_{3n-1},Y_{3n}),\\G_1(Y_{3n+1},Y_{3n},Y_{3n-1}),\\G_2(X_{3n-2},X_{3n-1},X_{3n})\end{matrix}\right\}\end{aligned}$$

$$G_1(X_{3n+1},X_{3n},X_{3n-1})\le C\max\left\{\begin{matrix}G_1(Y_{3n-2},Y_{3n-1},Y_{3n}),\\ \\G_2(X_{3n-2},X_{3n-1},X_{3n})\end{matrix}\right\}$$

Therefore, from the above inequalities, we obtain

$$\begin{aligned}G_1(X_{n+1},X_n,X_{n-1})&=C^n\max\ \{G_1(x,X_1,X_2),G_2(Y_1,Y_2,Y_3)\}\\G_2(Y_{n+1},Y_n,Y_{n-1})&=C^n\max\ \{G_1(x,X_1,X_2),G_2(Y_1,Y_2,Y_3)\}\end{aligned}$$

Let $P=\max\ \{G_1(x,X_1,X_2),G_2(Y_1,Y_2,Y_3)\}$

$$\begin{aligned}G_1(X_{n+1},X_n,X_{n-1})&=C^n\,P\\G_2(Y_{n+1},Y_n,Y_{n-1})&=C^n\,P\end{aligned}$$

Now, for all l,m,n with $l>m>n$,

$$\begin{aligned}&G_1(X_n,X_m,X_l)\\&\le G_1(X_n,X_{n+1},X_{n+1})+G(X_{n+1},X_{n+2},X_{n+2})+\cdots\\&\quad+G_1(X_{l-1},X_{l-1},X_l)\\&\le G_1(X_n,X_{n+1},X_{n+2})+G(X_{n+1},X_{n+2},X_{n+3})+\cdots\\&\quad+G_1(X_{l-2},X_{l-1},X_l)\\&\le C^n\,P+C^{n+1}P+\cdots+C^{l-2}\,P\\&\le C^n\,P(1+C+\cdots+C^{l-n-2})\\&\le\frac{C^nP}{1-C}\end{aligned}$$

Also if $l=m>n$ and $l>m=n$, we obtain

$$G_1(x,x_m,x_l)\longrightarrow 0\text{ as }n,m,l\longrightarrow\infty.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in $G_1 -$ metric space X. By $G_1 -$ completeness of X, there exists $z \in X$ such that $\{x_n\}$ converge to z as $n \longrightarrow \infty$.

i.e., $G_1(z, x_n, x_m) \longrightarrow 0$ as $n \longrightarrow \infty$.

Similarly, $\{y_n\}$ is a Cauchy sequence in $G_2 -$ metric space Y. By $G_2 -$ completeness of Y, there exists $w \in Y$ such that $\{y_n\}$ converge to w as $n \longrightarrow \infty$.

i.e., $G_2(w, y_n, y_m) \longrightarrow 0$ as $n \longrightarrow \infty$.

Now,

$$\begin{aligned}
G_1(S_2T_2x_{3n-1}, z, z) \le\ & G_1(S_2T_2x_{3n-1}, x_{3n-1}, x_{3n-1}) \\
& + G_1(x_{3n-1}, z, z) \\
\le\ & G_1(S_2T_2x_{3n-1}, S_1T_1x_{3n-2}, S_1T_1x_{3n-2}) \\
& + G_1(x_{3n-1}, z, z) \\
\le\ & C \max\left\{\begin{matrix} G_1(x_{3n-2}, x_{3n-2}, x_{3n-1}), \\ G_1(x_{3n-2}, x_{3n-1}, x_{3n-1}), \\ G_1(x_{3n-2}, x_{3n-1}, x_{3n-2}), \\ G_1(x_{3n-1}, x_{3n}, x_{3n-2}), \\ G_2(T_2x_{3n-1}, T_1x_{3n-2}, T_1x_{3n-2}) \end{matrix}\right\} \\
& + G_1(x_{3n-1}, z, z) \\
\le\ & C \max\left\{\begin{matrix} G_1(x_{3n-2}, x_{3n-2}, x_{3n-1}), \\ G_1(x_{3n-2}, x_{3n-1}, x_{3n-1}), \\ G_1(x_{3n-2}, x_{3n-1}, x_{3n-2}), \\ G_1(x_{3n-1}, x_{3n}, x_{3n-2}), \\ G_2(y_{3n-1}, y_{3n-2}, y_{3n}) \end{matrix}\right\} \\
& + G_1(x_{3n-1}, z, z) \\
& \to 0 \ as\ n \to \infty.
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} S_2T_2x_{3n-1} &= \{z\} = \lim_{n\to\infty} S_2y_{3n-1}. \\
\lim_{n\to\infty} S_1T_1x_{3n-2} &= \{z\} = \lim_{n\to\infty} S_1y_{3n-2}. \\
\lim_{n\to\infty} S_3T_3x_{3n} &= \{z\} = \lim_{n\to\infty} S_3y_{3n}. \\
\lim_{n\to\infty} T_2S_1y_{3n-2} &= \{w\} = \lim_{n\to\infty} T_2x_{3n-1}. \\
\lim_{n\to\infty} T_3S_2y_{3n-1} &= \{w\} = \lim_{n\to\infty} T_3x_{3n}. \\
\lim_{n\to\infty} T_1S_3y_{3n} &= \{w\} = \lim_{n\to\infty} T_1x_{3n+1}.
\end{aligned}$$

If T_1 is continuous, $\lim_{n\to\infty} T_1x_{3n+1} = T_1z = \{w\}$.

$$\begin{aligned}
& G_1(S_1T_1z, S_2T_2x_{3n-1}, S_3T_3x_{3n}) \\
& \le C \max\left\{\begin{matrix} G_1(z, x_{3n-1}, x_{3n}), \\ G_1(z, S_1T_1z, x_{3n}), \\ G_1(x_{3n-1}, x_{3n}, z), \\ G_1(x_{3n}, x_{3n+1}, x_{3n-1}), \\ G_2(T_1z, T_2x_{3n-1}, T_3x_{3n}) \end{matrix}\right\}
\end{aligned}$$

As $n \longrightarrow \infty$, we obtain

$$\begin{aligned}
& G_1(S_1T_1z, z, z) \le C\ G_1(z, S_1T_1z, x_{3n}) \\
\Rightarrow\ & G_1(S_1T_1z, z, z) = 0 \\
\Rightarrow\ & S_1T_1z = \{z\} = S_1w
\end{aligned}$$

By symmetry,

$$\begin{aligned}
S_2T_2z &= \{z\} = S_2w \\
S_3T_3z &= \{z\} = S_3w \\
T_2S_1w &= \{w\} = T_2z \\
T_3S_2w &= \{w\} = T_3z \\
T_1S_3w &= \{w\} = T_1z
\end{aligned}$$

We obtain from the above inequalities

$$\begin{aligned}
S_1T_1z = S_2T_2z = S_3T_3z = \{z\} = S_1w = S_2w = S_3w. \\
T_2S_1w = T_3S_2w = T_1S_3w = \{w\} = T_2z = T_3z = T_1z.
\end{aligned}$$

Hence, z is the common fixed point of T_1S_1, T_2S_2 and T_3S_3. Similarly, w is the common fixed point of T_2S_1, T_3S_2 and T_1S_3.

For the proof of uniqueness assume that, S_1T_1, S_2T_2 and S_3T_3 have a common fixed point z', such that

$$S_1T_1z' = z',\ S_2T_2z' = z' \text{ and } S_3T_3z' = z'.$$

We have

$$\begin{aligned}
G_1(S_1T_1z', S_2T_2z', S_3T_3z') \le\ & C \max\left\{\begin{matrix} G_1(z', z', z'), \\ G_1(z', S_1T_1z', z'), \\ G_1(z', S_2T_2z', z'), \\ G_1(z', S_3T_3z', z'), \\ G_2(T_1z', T_2z', T_3z') \end{matrix}\right\} \\
\le\ & C\ G_2(T_1z', T_2z', T_3z') \\
\le\ & C\ G_2(T_2S_1T_1z', T_3S_2T_2z', T_1S_3T_3z') \\
G_1(S_1T_1z', S_2T_2z', S_3T_3z') \le\ & C^2 \max\left\{\begin{matrix} G_2(T_1z', T_2z', T_3z'), \\ G_2(T_1z', T_2S_1T_1z', T_3z'), \\ G_2(T_2z', T_3S_2T_2z', T_1z'), \\ G_2(T_3z', T_1S_3T_3z', T_2z'), \\ G_1(S_1T_1z', S_2T_2z', S_3T_3z') \end{matrix}\right\} \\
\le\ & C^2\ G_1(S_1T_1z', S_2T_2z', S_3T_3z')
\end{aligned}$$

Which yields,

$$\begin{aligned}
& G_1(S_1T_1z', S_2T_2z', S_3T_3z') = 0 \\
& \Rightarrow S_1T_1z' = S_2T_2z' = S_3T_3z' \ as\ C < 1 \\
& \Rightarrow S_1T_1z' = S_2T_2z' = S_3T_3z' = \{z'\}
\end{aligned}$$

Thus, $T_1z' = T_2z' = T_3z' = \{w'\}$ (say)

Then,

$$\begin{aligned}
S_1T_1z' = S_2T_2z' = S_3T_3z' = \{z'\} = S_1w' = S_2w' = S_3w'. \\
T_2S_1w' = T_3S_2w' = T_1S_3w' = \{w'\} = T_2z' = T_3z' = T_1z'.
\end{aligned}$$

Now,

$$\begin{aligned}
G_1(z, z, z') =\ & G_1(S_1T_1z, S_2T_2z, S_3T_3z') \\
\le\ & C \max\left\{\begin{matrix} G_1(z, z, z'), G_1(z, S_1T_1z, z'), \\ G_1(z, S_2T_2z, z), G_1(z', S_3T_3z', z), \\ G_2(T_1z, T_2z, T_3z') \end{matrix}\right\} \\
\le\ & C \max\left\{\begin{matrix} G_1(z, z, z'), G_1(z, z, z'), \\ G_1(z, z, z), G_1(z', z', z), \\ G_2(T_1z, T_2z, T_3z') \end{matrix}\right\} \\
\le\ & C \max\left\{\begin{matrix} G_1(z, z, z'), G_1(z, z, z'), \\ G_1(z, z, z), G_1(z, z, z'), \\ G_2(T_1z, T_2z, T_3z') \end{matrix}\right\} \\
\le\ & C\ G_2(T_1z, T_2z, T_3z') \\
=\ & C\ G_2(T_3S_2w, T_1S_3w, T_2S_1w') \\
=\ & C\ G_2\ (T_2S_1w', T_3S_2w, T_1S_3w) \\
\le\ & C^2 \max\left\{\begin{matrix} G_2(w', w, w), \\ G_2(w', T_2S_1w', w), \\ G_2(w, T_3S_2w, w'), \\ G_2(w, T_1S_3w, w), \\ G_1(S_1w', S_2w, S_3w) \end{matrix}\right\} \\
\le\ & C^2 \max\left\{\begin{matrix} G_2(w', w, w), \\ G_2(w', w', w), \\ G_2(w, w, w'), \\ G_2(w, w, w), \\ G_1(z', z, z) \end{matrix}\right\} \\
\le\ & C^2 \max\ \{G_2(w', w, w), G_1(z', z, z)\}
\end{aligned}$$

Which yields,

$$\begin{aligned}
G_1(z, z, z')\} &= 0 \\
z &= z'
\end{aligned}$$

Thus,

Hence, z is the unique common fixed point of T_1S_1, T_2S_2 and T_3S_3. Similarly, w is the unique common fixed point of T_2S_1, T_3S_2 and T_1S_3. This completes the proof.

Example 3.4. Let $X = \left[0, \frac{1}{2}\right], Y = \left[\frac{1}{2}, 1\right]$ and

$$G_1(x,y,z) = G_2(x,y,z) = \begin{cases} 0 & , \ if\ x = y = z \\ \max\{x,y,z\}, & otherwise \end{cases}$$

. Then (X, G_1) and (Y, G_2) are symmetric G-metric spaces. Define $T_1, T_2, T_3 : X \to B(Y)$ by

$$T_1(x) = \begin{cases} x^4 - 2x^3 + \frac{3}{2}x^2 + \frac{x}{2} + \frac{9}{16} , & if\ x\ is\ rational \\ 0, & otherwise \end{cases}$$

$$T_2(x) = \begin{cases} x^2 + \frac{3}{2}, & if\ x\ is\ rational \\ 0 , & otherwise \end{cases}$$

$$T_3(x) = x + \frac{1}{2}$$

Define $S_1, S_2, S_3 : Y \to B(X)$ by

$$S_1(x) = \begin{cases} x - \frac{1}{2} , & if\ x\ is\ rational \\ 0, & otherwise \end{cases}$$

$$S_2(x) = \begin{cases} x^4 - 4x^3 + 6x^2 - 3x + \frac{1}{2} , & if\ x\ is\ rational \\ 0 , & otherwise \end{cases}$$

$$S_3(x) = \begin{cases} x^2 - x + \frac{1}{2}, & if\ x\ is\ rational \\ 0 , & otherwise \end{cases}$$

Then,

$$S_1T_1\left(\frac{1}{2}\right) = S_1\left(\frac{1}{16} - \frac{2}{8} + \frac{3}{8} + \frac{1}{4} + \frac{9}{16}\right) = S_1(1) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$S_1T_1\left(\frac{1}{2}\right) = S_1(1) = \frac{1}{2}.$$

$$S_2T_2\left(\frac{1}{2}\right) = S_2\left(\frac{1}{4} + \frac{3}{4}\right) = S_2(1) = 1 - 4 + 6 - 3 + \frac{1}{2} = \frac{1}{2}$$

$$S_2T_2\left(\frac{1}{2}\right) = S_2(1) = \frac{1}{2}.$$

$$S_3T_3\left(\frac{1}{2}\right) = S_3\left(\frac{1}{2} + \frac{1}{2}\right) = S_3(1) = 1 - 1 + \frac{1}{2} = \frac{1}{2}$$

$$S_3T_3\left(\frac{1}{2}\right) = S_3(1) = \frac{1}{2}.$$

Therefore, we have

$$S_1T_1\left(\frac{1}{2}\right) = S_2T_2\left(\frac{1}{2}\right) = S_3T_3\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right) = S_1(1) = S_2(1) = S_3(1).$$

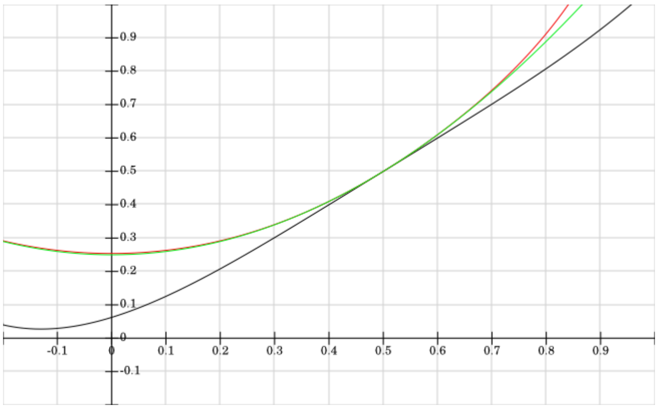
Also,

$$T_2S_1(1) = T_2\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{3}{4} = 1$$

$$T_2S_1(1) = T_2\left(\frac{1}{2}\right) = 1$$

$$T_3S_2(1) = T_3\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

$$S_1T_1\left(\frac{1}{2}\right) = S_2T_2\left(\frac{1}{2}\right) = S_3T_3\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)$$



$$T_3S_2(1) = T_3\left(\frac{1}{2}\right) = 1$$

$$T_1S_3(1) = T_1\left(\frac{1}{2}\right) = \frac{1}{16} - \frac{2}{8} + \frac{3}{8} + \frac{1}{4} + \frac{9}{16} = 1$$

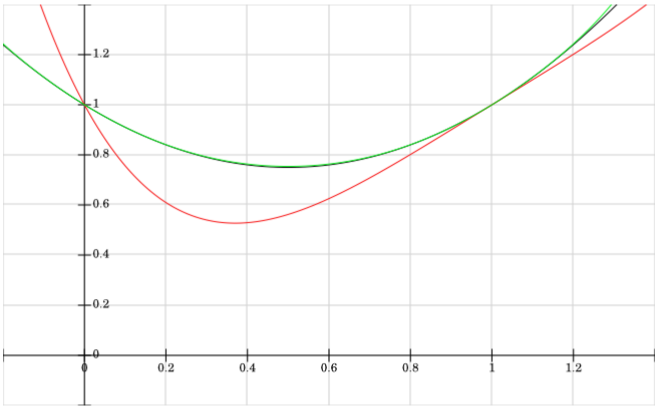
$$T_1S_3(1) = T_1\left(\frac{1}{2}\right) = 1$$

From the above equations, we have

$$T_2S_1(1) = T_3S_2(1) = T_1S_3(1) = 1 = T_1\left(\frac{1}{2}\right) = T_2\left(\frac{1}{2}\right) = T_3\left(\frac{1}{2}\right).$$

Hence $\frac{1}{2}$ is the unique common fixed point of S_1T_1, S_2T_2 and S_3T_3 and 1 is the unique common fixed point of T_2S_1, T_3S_2 and T_1S_3.

Graphical View:



$T_2S_1(1) = T_3S_2(1) = T_1S_3(1) = 1$

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