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On the Construction of Variance Balanced Designs

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ABSTRACT

In this paper an attempt is made to obtain construction methods of variance balanced (VB) designs. The obtained methods are constructed using balanced incomplete block (BIB) designs. The examples are given to show how they can be applied.

1. Introduction

Many researchers have contributed in the construction of non- equireplicate and non-proper VB designs by understanding their significance. John (1964) constructed binary and ternary VB designs using a BIB design. Kulshreshtha, et al. (1972) generalized the method of John (1964). Federer (1961) introduced augmented designs. Das (1958), explaining the significance, introduced reinforced incomplete block designs and gave reinforced BIB design. As a significant contribution, Das and Ghosh (1985) introduced general efficiency balanced (GEB) designs unifying BIB designs, VB and EB designs and constructed new VB and EB designs using the technique of reinforcement. Kageyama and Mukerjee (1986) gave characterization of construction of GEB designs, non-existence theorem and unified method of construction of GEB designs. Here we give new methods for the construction of VB designs by using BIB designs.

Following are the notations which have been used for the construction of new methods:

$\mathbf{1}_p$: $p \times 1$ column vector of ones

$\mathbf{1}_p'$: $1 \times p$ row vector of ones,

$\otimes$: kronecker product of matrices,

$\mathbf{1}_p' \otimes N$: p replications of N,

I_p: identity matrix of order p,

$O_{p\times q}$: null matrix of order $p \times q$,

$J_{p\times q}$: matrix of ones of order $p \times q$

$p_1, p_2, x_l (l = 1,2 \dots,7)$: the positive integers.

2. Methods of Construction of VB Designs

In this section, we describe construction methods of binary and non-binary VB designs by using some known BIB designs.

Theorem 2.1: Let N_1 and N_2 be the incidence matrices of two BIB designs D_1 and D_2 with parameters v_1, b_1, r_1, k_1, λ_1 and v_2, b_2, r_2, k_2, λ_2 respectively. Then

$$N = \begin{bmatrix} x_1\mathbf{1}_{p_1}' \otimes N_1 & O_{v_1\times p_2 b_2} & x_4 J_{v_1\times v_2} \\ x_2 J_{v_2\times p_1 b_1} & x_3\mathbf{1}_{p_2}' \otimes N_2 & x_5 I_{v_2} \end{bmatrix}$$

is the incidence matrix of a VB design D with parameters $v = v_1 + v_2$, $b = p_1 b_1 + p_2 b_2 + v_2$, $\boldsymbol{r}' = \{ F_1\mathbf{1}_{v_1}', F_2\mathbf{1}_{v_2}'\}$, $\boldsymbol{k}' = \{(x_1k_1 +$ *x2v2***1***p1b1', x3k2***1***p2b2', x4v1+x5***1***v2'* such that

$$\frac{p_1 {x_1}^2 \lambda_1}{(x_1k_1 + x_2v_2)} + \frac{{x_4}^2 v_2}{(x_4v_1 + x_5)} = \frac{p_1x_1x_2r_1}{(x_1k_1 + x_2v_2)} + \frac{x_4x_5}{(x_4v_1 + x_5)}$$

$$= \frac{p_1 {x_2}^2 b_1}{(x_1k_1 + x_2v_2)} + \frac{p_2x_3\lambda_2}{k_2} \qquad (2.1)$$

where $F_1 = p_1x_1r_1 + x_4v_2$ and $F_2 = p_1x_2b_1 + p_2x_3r_2 + x_5$.

Proof: The off-diagonal elements of the $C = (c_{ij})$ matrix of D are:

$$c_{ij} = \frac{p_1 x_1{}^2 \lambda_1}{(x_1 k_1 + x_2 v_2)} + \frac{x_4{}^2 v_2}{(x_4 v_1 + x_5)} \qquad ; i, j \le v_1 \ \& \ i \ne j \qquad (2.2)$$

$$c_{ij} = \frac{p_1 x_1 x_2 r_1}{(x_1 k_1 + x_2 v_2)} + \frac{x_4 x_5}{(x_4 v_1 + x_5)} \qquad ; i \le v_1 \ \& \ j \ge (v_1 + 1) \qquad (2.3)$$

$$c_{ij} = \frac{p_1 x_2{}^2 b_1}{(x_1 k_1 + x_2 v_2)} + \frac{p_2 x_3{}^2 \lambda_2}{x_3 k_2} \qquad ; i, j \ge (v_1 + 1) \ \& \ i \ne j \qquad (2.4)$$

By Rao (1958), we get the required result (2.1) upon equating (2.2), (2.3) and (2.4).

Example 2.1: Consider two BIB designs D_1 and D_2 having parameters $v_1 = 5$, $b_1 = 10$, $r_1 = 4$, $k_1 = 2$, $\lambda_1 = 1$ and $v_2 = 4$, $b_2 = 6$, $r_2 = 3$, $k_2 = 2$, $\lambda_2 = 1$ respectively. Blocks structure are: D_1(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5); D_2(6,7), (6,8), (6,9), (7,8), (7,9), (8,9). Then taking $p_1 = p_2 = x_1 = x_2 = x_3 = 1$, $x_4 = 4$ and $x_5 = 12$, the incidence matrix N of the obtained design is given by

$$N = \begin{bmatrix} 1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,04\,04\,04\,04 \\ 1\,0\,0\,0\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,04\,04\,04\,04 \\ 0\,1\,0\,0\,1\,0\,0\,1\,1\,0\,0\,0\,0\,0\,0\,0\,04\,04\,04\,04 \\ 0\,0\,1\,0\,0\,1\,0\,1\,0\,1\,0\,0\,0\,0\,0\,0\,04\,04\,04\,04 \\ 0\,0\,0\,1\,0\,0\,1\,0\,1\,1\,0\,0\,0\,0\,0\,0\,04\,04\,04\,04 \\ 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0\,12\,00\,00\,00 \\ 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,0\,0\,1\,1\,0\,00\,12\,00\,00 \\ 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,0\,1\,0\,1\,0\,1\,00\,00\,12\,00 \\ 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,0\,0\,1\,0\,1\,1\,00\,00\,00\,12 \end{bmatrix}$$

and C matrix of the obtained design is given by

$$C = \begin{bmatrix} \frac{52}{3} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} \\ \frac{-13}{6} & \frac{52}{3} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} \\ \frac{-13}{6} & \frac{-13}{6} & \frac{52}{3} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} \\ \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{52}{3} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} \\ \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{52}{3} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} \\ \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{52}{3} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} \\ \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{52}{3} & \frac{-13}{6} & \frac{-13}{6} \\ \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{52}{3} & \frac{-13}{6} \\ \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{-13}{6} & \frac{52}{3} \end{bmatrix}$$

The parameters of the four-ary VB design obtained are $v = 9$, $b = 20$, $\boldsymbol{r}' = \{20\mathbf{1}_5', 25\mathbf{1}_4'\}$, $\boldsymbol{k}' = \{6\mathbf{1}_{10}', 2\mathbf{1}_6', 32\mathbf{1}_4'\}$.

Theorem 2.2: Let N_1 and N_2 be the incidence matrices of two BIB designs D_1 and D_2 with parameters v_1, b_1, r_1, k_1, λ_1 and v_2, b_2, r_2, k_2, λ_2 respectively such that $v_1 = v_2$. Then

$$N = \begin{bmatrix} x_1 \mathbf{1}_{p_1}' \otimes N_1 & x_3 \mathbf{1}_{v_1}' \otimes N_2 & x_5 \mathbf{1}_{p_2}' \otimes N_1 & x_6 \mathbf{1}_{v_1}' \otimes I_{v_1} \\ x_2 J_{v_1 \times p_1 b_1} & x_4 I_{v_1} \otimes \mathbf{1}_{b_2}' & O_{v_1 \times p_2 b_1} & x_7 I_{v_1} \otimes \mathbf{1}_{v_1}' \end{bmatrix}$$

is the incidence matrix of a VB design D with parameters $v = v_1 + v_1 = 2v_1$, $b = p_1 b_1 + v_1 b_2 + p_2 b_1 + v_1{}^2$, $\boldsymbol{r}' = \{F_3 \mathbf{1}_{v_1}', F_4 \mathbf{1}_{v_1}'\}$, $\boldsymbol{k}' = \left\{(x_1 k_1 + x_2 v_1)\mathbf{1}_{p_1 b_1}', (x_3 k_2 + x_4)\mathbf{1}_{v_1 b_2}', \ x_5 k_1 \mathbf{1}_{p_2 b_1}', (x_6 + x_7)\mathbf{1}_{v_1{}^2}'\right\}$

such that

$$\frac{p_1 x_1{}^2 \lambda_1}{(x_1 k_1 + x_2 v_1)} + \frac{x_3{}^2 v_1 \lambda_2}{(x_3 k_2 + x_4)} + \frac{p_2 x_5 \lambda_1}{k_1} = \frac{p_1 x_1 x_2 r_1}{(x_1 k_1 + x_2 v_1)} + \frac{x_3 x_4 r_2}{(x_3 k_2 + x_4)} + \frac{x_6 x_7}{(x_6 + x_7)}$$

$$= \frac{p_1 x_2^{\,2} b_1}{(x_1 k_1 + x_2 v_1)} \tag{2.5}$$

where $F_3 = p_1 x_1 r_1 + x_3 v_1 r_2 + p_2 x_5 r_1 + x_6 v_1$ and $F_4 = p_1 x_2 b_1 + x_4 b_2 + x_7 v_1$.

Proof: The off-diagonal elements of the $C = (c_{ij})$ matrix of D are:

$$c_{ij} = \frac{p_1 x_1^{\,2} \lambda_1}{(x_1 k_1 + x_2 v_1)} + \frac{x_3^{\,2} v_1 \lambda_2}{(x_3 k_2 + x_4)} + \frac{p_2 x_5 \lambda_1}{k_1} \qquad ; i,j \le v_1 \ \& \ i \ne j \tag{2.6}$$

$$c_{ij} = \frac{p_1 x_1 x_2 r_1}{(x_1 k_1 + x_2 v_1)} + \frac{x_3 x_4 r_2}{(x_3 k_2 + x_4)} + \frac{x_6 x_7}{(x_6 + x_7)} \qquad ; i \le v_1 \ \& \ j \ge (v_1 + 1) \tag{2.7}$$

$$c_{ij} = \frac{p_1 x_2^{\,2} b_1}{(x_1 k_1 + x_2 v_1)} \qquad ; i,j \ge (v_1 + 1) \ \& \ i \ne j \tag{2.8}$$

By Rao (1958), we get the required result (2.5) upon equating (2.6), (2.7) and (2.8).

Example 2.2: Consider two BIB designs D_1 and D_2 having parameters $v_1 = 4$, $b_1 = 6$, $r_1 = 3$, $k_1 = 2$, $\lambda_1 = 1$ and $v_2 = 4$, $b_2 = 4$, $r_2 = 3$, $k_2 = 3$, $\lambda_2 = 2$ respectively. Blocks structure are: D_1(1,2), (1,3), (1,4), (2,3), (2,4), (3,4); D_2(5,6,7), (5,6,8), (5,7,8), (6,7,8). Then taking $p_1 = x_7 = 3$ and $p_2 = x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 1$, the incidence matrix N of the obtained design is given by

$$N = \left[\begin{array}{cccccccccccccccccccccccccccccccccccccccc}
1&1&1&0&0&0&1&1&1&0&0&0&1&1&1&0&0&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&0&0\\
1&0&0&1&1&0&1&0&0&1&1&0&1&0&0&1&1&0&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&0&0&1&1&0\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&0&1&0&1&0&1\\
0&0&1&0&1&1&0&0&1&0&1&1&0&0&1&0&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&0&1&0&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0
\end{array}\right.$$

$$\left.\begin{array}{cccccccccccccccc}
1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0\\
0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\
3&3&3&3&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&3&3&3&3&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&3&3&3&3&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&3&3&3&3
\end{array}\right]$$

and C matrix of the obtained design is given by

$$C = \begin{bmatrix}
21&3&3&3&3&3&3&3\\
3&21&3&3&3&3&3&3\\
3&3&21&3&3&3&3&3\\
3&3&3&21&3&3&3&3\\
3&3&3&3&21&3&3&3\\
3&3&3&3&3&21&3&3\\
3&3&3&3&3&3&21&3\\
3&3&3&3&3&3&3&21
\end{bmatrix}$$

The parameters of the ternary VB design obtained are $v = 8$, $b = 56$, $\boldsymbol{r}' = \{28\mathbf{1}_4', 34\mathbf{1}_4'\}$, $\boldsymbol{k}' = \{6\mathbf{1}_{18}', 4\mathbf{1}_{16}', 2\mathbf{1}_{6}', 4\mathbf{1}_{16}'\}$.

3. Conclusion

Here variance balanced designs are obtained by the new construction methods. The methods are based on incidence matrices of BIB designs. The obtained designs can be applicable in industrial, pharmaceutical and agricultural experiments.

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