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$\mathcal{G}$-Connectedness in $\mathcal{G}$-Topological Simple Groups

*1C. Selvi; 2R. Selvi & 3K. Shakila

*1Research Scholar, Sri Parasakthi College for Women affiliated to Manonmaniam Sudaranar University (India)
2Assistant Professor, Sri Parasakthi College for Women affiliated to Manonmaniam Sudaranar University (India)
3Research Scholar, Sri Parasakthi College for Women affiliated to Manonmaniam Sudaranar University (India)

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***Corresponding Author**
Email: meselvi89[at]rediffmail.com

ABSTRACT

$\mathcal{G}$-connectedness for $\mathcal{G}$-topological group was already introduced in $\mathcal{G}$-topological space. In this paper we investigate $\mathcal{G}$-connectedness for $\mathcal{G}$-topological simple group. Also some basic results and theorems are studied.

1. Introduction

The concept of generalized topological space introduced by A.csaszar in 2002[6]. In[4] we defined the $\mathcal{G}$-topological simple group structure and we proved some basic results.In[3] Murad hussain et.al. introduced the concept $\mathcal{G}$-connectedness for $\mathcal{G}$-topological group. Also they were studied $\mathcal{G}$-connectedness in quotients of $\mathcal{G}$-topological group.

In this paper we have to investigate the concept of $\mathcal{G}$-connectedness for $\mathcal{G}$-topological simple group. Also we were studied the some basic results and theorem.

2. Preliminaries

Definition: 2.1[3] Let X be a $\mathcal{G}$-topological space and let $U, V \subset X$. Then we say that the pair U, V is $\mathcal{G}$-separated if $Cl_{\mathcal{G}}(U) \cap V = Cl_{\mathcal{G}}(V) \cap U = \phi$.

Definition: 2.2[3] Let X be a $\mathcal{G}$-topological space. A set $S \subset X$ is $\mathcal{G}$-connected if there are two non-empty $\mathcal{G}$-separated sets U and V such that $U \cup V = S$.

Definition: 2.3[3] Let X be a $\mathcal{G}$-topological space. Then the space X itself is $\mathcal{G}$-connected if it is a $\mathcal{G}$-connected subset of itself.

Lemma: 2.4[3] Let $f: X \to Y$ be a $\mathcal{G}$-continuous mapping between $\mathcal{G}$-topological spaces. If X is $\mathcal{G}$-connected, so is $f(X)$.

Definition: 2.5[3] Let $f: X \to Y$ be a $\mathcal{G}$-open and injective mapping between $\mathcal{G}$-topological spaces and let $S \subset X$ if $f(S)$ is $\mathcal{G}$-connected, so is S.

Definition: 2.6[3] Let X be a $\mathcal{G}$-topological space, $A \subset X$. For $x \in A$, the set

$$A_x = \bigcup_{x \in S \subset A} S,$$

Where S is $\mathcal{G}$-connected in A, is called the $\mathcal{G}$-component of A belonging to x.

Definition: 2.7[2] Let G be a $\mathcal{G}$-topological group. Then a subgroup K is called discrete group if $K \cap V$ is atmost one singleton set, for any $\mathcal{G}$-open set V in G.

Definition: 2.8[14] A group G is called a Simple group if it has no non trivial normal subgroup.

Definition: 2.9[4] A $\mathcal{G}$-topological simple group G is a simple group which is also a $\mathcal{G}$-topological space if the following conditions are satisfied.

(i). The multiplication mapping $m: G \times G \longrightarrow G$ defined by $m(x, y) = x * y,\ x, y \in G$ is $\mathcal{G}$- continuous.

(ii). The inverse mapping $i: G \longrightarrow G$ defined by $i(x) = x^{-1}, x \in G$ is $\mathcal{G}$-continuous.

Note: **2.10 [4]** Let G be a $\mathcal{G}$-topological simple group and H be a normal subgroup of G. Here H is either proper trivial normal subgroup of G or improper trivial normal subgroup of G. Let ϕ be a mapping from $G \to \frac{G}{H}$ by $\phi(x) = xH, \forall\, x \in G$. Now we can define a $\mathcal{G}$-topology on $\frac{G}{H}$, U is $\mathcal{G}$-open in $\frac{G}{H}$ $\Leftrightarrow \phi^{-1}(U)$ is $\mathcal{G}$-open in G.

3. $\mathcal{G}$-CONNECTEDNESS IN $\mathcal{G}$-TOPOLOGICAL SIMPLE GROUPS

Theorem: 3.1 For any two $\mathcal{G}$-connected subsets E and F of a $\mathcal{G}$-topological simple group G, their product EF in G is a $\mathcal{G}$-connected subspace of G.

Proof: Since the multiplication is $\mathcal{G}$-continuous, the subspace EF of G is a $\mathcal{G}$-continuous image of the Cartesian product $E \times F$ of the spaces E and F. Since $E \times F$ is $\mathcal{G}$-connected, the space EF is $\mathcal{G}$-connected.

Lemma: 3.2[4] Let G be $\mathcal{G}$-topological simple group and U be $\mathcal{G}$-open subset of G , F is closed in G and A be any subset of G. Then

(i). aU,Ua and AU, UA are $\mathcal{G}$-open in G.

(ii). aF and Fa are $\mathcal{G}$-closed in G.

Lemma: 3.3[4] Every $\mathcal{G}$-open subgroup H of $\mathcal{G}$-topological simple group G is $\mathcal{G}$-closed in $G\cdot$

Theorem: 3.4 Let U be an arbitrary $\mathcal{G}$-open neighbourhood of the neutral element e of a $\mathcal{G}$-connected simple topological group G. Then $G = \cup_{n=1}^{\infty} U^n$.

Proof: Choose a $\mathcal{G}$-open symmetric neighbourhood V of e in G such that $V \subset U$. By induction on n and by lemma 3.2, for every positive integer n, V^n is $\mathcal{G}$-open. Let $H = \cup_{n=1}^{\infty} V^n$. Hence $H = \cup_{n=1}^{\infty} V^n$ is $\mathcal{G}$-open. Let. Now let $x \in V^p, y \in V^q$ are the elements from H. Then ,

$x * y \in V^{p+q} \subset H$

$\Longrightarrow x * y \in H.$

Take an element $x \in H$ and $x \in V^k$. Then $x^{-1} \in (V^{-1})^k = V^k \in H$. Therefore H is a subgroup of G. By lemma 3.3, H is also a $\mathcal{G}$-closed in G. Since G is $\mathcal{G}$-connected, and H is non empty and both $\mathcal{G}$-closed and $\mathcal{G}$-open, $G = H$. As $V \subset U$, it follows that $G = \cup_{n=1}^{\infty} U^n$.

Theorem: 3.5 Let G be a $\mathcal{G}$-connected $\mathcal{G}$-topological simple group and e its identity element. If U is an $\mathcal{G}$-open neighbourhood of e, then G is generated by U.

Proof: Let U be a $\mathcal{G}$-open neighbourhood of e. For each $n \in \mathbb{N}$, we denote by U^n the set of elements of the form $u_1 u_2 \dots u_n$, where $u_i \in U$. Let $= \cup_{n \in \mathbb{N}} U^n$ **.** Since each U^n is $\mathcal{G}$-open, we have that W is a $\mathcal{G}$-open, W is also a $\mathcal{G}$-closed by lemma 3.3. Let g be an element of generalized closure of W. That is $g \in Cl_{\mathcal{G}}(W)$. Since gU^{-1} is a $\mathcal{G}$-open neighbourhood of g, it must intersects W. Thus, let $h \in W \cap gU^{-1}$. Since $h \in gU^{-1}$, then $h = gu^{-1}$ for some elements $u \in U$. Since $h \in W$, then $h \in U^n$ for some $n \in \mathbb{N}$. So $h = u_1 u_2 \dots u_n$ with each $u_i \in U$.

We have $g = hu$ for some $u \in U$.

$= u_1 u_2 \dots u_n u$ for some $u \in U$.

This implies that $g \in U^{n+1} \subseteq W$. Hence W is $\mathcal{G}$-closed. Since G is $\mathcal{G}$-connected and W is $\mathcal{G}$-open and $\mathcal{G}$-closed, we must have $W = G$. This means that G is generated by U.

Theorem: 3.6 Let K be a discrete invariant subgroup of a $\mathcal{G}$-connected $\mathcal{G}$-topological simple abelian group G. If for any $\mathcal{G}$-open neighbourhood U of x in G there exists a $\mathcal{G}$-open symmetric neighbourhood V of e in G such that $VxV \subset U$, then every element of K commutes with every element of G. i.e. K is contained in the center of the simple abelian group G.

Proof: Since G is a $\mathcal{G}$-topological simple abelian group, G and $\{e\}$ are the only invariant subgroup of G. Therefore the proof is trivial.

Lemma: 3.7[3] The $\mathcal{G}$-component of a $\mathcal{G}$-closed set A in $\mathcal{G}$-topological space X is again $\mathcal{G}$-closed.

Lemma: 3.8[4] Let $\frac{G}{H}$ be a $\mathcal{G}$-topological simple group with quotient $\mathcal{G}$-topology and $\phi : G \to \frac{G}{H}$ by $\phi(x) = xH, \forall x \in G$. Then the following statement is hold.

(i). ϕ is onto.

(ii). ϕ is $\mathcal{G}$-continuous.

(iii). ϕ is $\mathcal{G}$-open.

(iv). ϕ is homomorphism.

Theorem: 3.9 Let G be a $\mathcal{G}$-topological simple group and H is a $\mathcal{G}$-connected subgroup of G. Then

(i). If $\frac{G}{H}$ is $\mathcal{G}$-connected, then G is $\mathcal{G}$-connected.

(ii). If H is a $\mathcal{G}$-closed invariant subgroup of G and $\frac{G}{H}$ is $\mathcal{G}$-connected, then G is $\mathcal{G}$-connected.

(iii). The $\mathcal{G}$-connected component containing e is a $\mathcal{G}$-closed invariant subgroup of G, where e is the identity element of G.

(iv). If H is a $\mathcal{G}$-dense subgroup of a $\mathcal{G}$-connected $\mathcal{G}$-topological simple group, then every $\mathcal{G}$-neighbourhood U of the identity element in H algebraically generates the group H.

Proof: (i). Suppose G is not $\mathcal{G}$-connected. Then there exists two disjoint non empty $\mathcal{G}$-open set U, V such that $U \cup V = G$. Denoting by π the map $x \to xH$, we see that $\frac{G}{H} = \pi(U) \cup \pi(V)$. By above lemma π is a $\mathcal{G}$-open mapping and $\frac{G}{H}$ is the union of two nonempty $\mathcal{G}$-open sets $\pi(U)$ and $\pi(V)$. Now the assumption of $\mathcal{G}$-connectedness of $\frac{G}{H}$ implies that $\pi(U)$ and $\pi(V)$ have some common point, zH. But disjointness assumption on U and V means that, for this to happen there must exists $x \in$ U and $y \in V$ such that $x^{-1}y \in H$. So the coset xH intersects both U and V in G. As $U \cup V = G$ provides a disconnection xH has to be disconnected. But xH is homeomorphic to H. So this contradicts the hypothesis that H is $\mathcal{G}$-connected. So G is $\mathcal{G}$-connected.

(ii) . Suppose that H and $\frac{G}{H}$ are $\mathcal{G}$-connected and $f: G \to \{0,1\}$ be an arbitrary $\mathcal{G}$-continuous map. The restriction of f to H must be constant and since each coset gH is $\mathcal{G}$-connected, f must be constant on gH as well taking value $f(g)$. Thus we have a well defined map $\tilde{f}: \frac{G}{H} \to \{0,1\}$ such that $\tilde{f}\ o\ \pi = f$. By the fundamental property of quotient spaces, it follows that $\tilde{f}$ is $\mathcal{G}$-continuous and so must be constant. Since $\frac{G}{H}$ is $\mathcal{G}$-connected. Hence, f is also constant and we conclude that G is $\mathcal{G}$-connected.

(iii). Let F be the $\mathcal{G}$-component of the identity e. By above lemma, F is $\mathcal{G}$-closed. Let $a \in F$. Since the multiplication and inversion mappings in G are $\mathcal{G}$-continuous, by lemma 2.4. The set aF^{-1} is also $\mathcal{G}$-connected, and since $e \in aF^{-1}$ we have $aF^{-1} \subset F$. Hence, for every $b \in F$ we have $ab^{-1} \in F$. i.e. F is a subgroup of G. If g is any element of $G,$ then $L_{g^{-1}}\ o\ R_g = g^{-1}Fg$ is a $\mathcal{G}$-connected subset containing e. Since F is a $\mathcal{G}$-component, $g^{-1}Fg \subset F$ for every $g \in G$. i.e, F is an invariant subgroup of G.

(iv). Proof follows from theorem 3.5

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