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Construction of Balanced Bipartite Block Designs with Unequal Block Sizes

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ABSTRACT

In this paper some methods of construction of balanced bipartite block (BBPB) designs are introduced for comparison of two disjoint sets of treatments (test treatments and control treatments). The methods are based on incidence matrices of balanced incomplete block (BIB) designs. The derived results are given with examples.

1. Introduction

Many researchers have contributed in the construction of incomplete block designs for comparison of test-control treatments by understanding their significance. Bechhofer and Tamhane (1981) introduced balanced treatment incomplete block (BTIB) designs. Kageyama and Sinha (1988) (see also Sinha and Kageyama (1990)) defined balanced bipartite block (BBPB) designs as an extension of these BTIB designs and gave some systematic methods of construction of BBPB designs. Majumdar (1986) obtained sufficient conditions for a block design to be A-optimal for comparison of a set of test treatments to a set of control treatments. Angelis and Moyssiadis (1991) (see also Angelis, Moyssiadis and Kageyama (1993)) gave balanced treatment incomplete block designs with unequal block sizes (BTIUB) for comparison of a set of test treatments to a single control treatment with unequal blocks. The more results have been studied by several researchers (see e.g. Jacroux (1992, 2000, 2002), Parsad and Gupta (1994), Parsad, Gupta and Singh (1996), Gupta and Parsad (2001)) for the comparison of test treatments and control treatments.

Here the problem of deriving an incomplete block design is considered for comparing v_1 test treatments with $v_2\,(\geq 2)$ control treatments. The v_1 test treatments are denoted by $1, 2, \dots, v_1$ and v_2 control treatments are denoted by $v_1 + 1, \dots, v_1 + v_2 (= v)$. Consider an incomplete block design $D\{v = (v_1 + v_2), b = \sum_{L=1}^{p} b_L, \boldsymbol{r}' = (r_1 \mathbf{1}'_{v_1}, r_2 \mathbf{1}'_{v_2}), \boldsymbol{k}' = (k_1 \mathbf{1}'_{b_1}, \dots, k_p \mathbf{1}'_{b_p})\}$ whose incidence matrix $N = (n_{ij})$ is of order $v(= v_1 + v2 \times b$. Then a connected block design is called a balanced bipartite block design with unequal block sizes (BBPBUB) of Jaggi, Parsad and Gupta (1999), if the information matrix for treatment effects i.e. the C matrix of the design is given by

$$C = \begin{bmatrix} (a_1 + s_1) I_{v_2} - s_1 J_{v_2} & -s_0 J_{v_2 \times v_3} \\ -s_0 J_{v_3 \times v_2} & (a_2 + s_2) I_{v_3} - s_2 J_{v_2} \end{bmatrix} \qquad (1.1)$$

where s_1, s_0 and s_2 are off-diagonal elements and a_1 and a_2 are some scalar constants such that $a_1 - (v_1 - 1)s_1 - v_2 s_0 = 0$ and $a_2 - (v_2 - 1)s_2 - v_1 s_0 = 0$. For a binary design, the constants a_1 and a_2 represent the functions of replications of the two sets of treatments and block sizes in the design.

In following section, we obtain some methods of construction of BBPBUB designs for comparison of a set of test treatments to a set of control treatments by using BIB designs. The definition of BIB design is given in Dey (2010).

In what follows, we denote by $\otimes$ the kronecker product of matrices, $O_{x \times y}$ the null matrix of order $x \times y$, I_x the identity matrix of order x, $J_{x \times y}$ the matrix of ones of order $x \times y$, $\mathbf{1}'_x$ the $1 \times x$ row vector of ones $\mathbf{1}'_x \otimes N$ the x replications of N and by p_1, p_2, p_3, p_4, p_5 the positive integers.

2. Methods of construction of BBPBUB Designs

Using the incidence matrices of BIB designs, etc. we describe below some methods of construction of BBPBUB designs.

Let N_L ($L = 1,2,3,4,5$) be the $v_L \times b_L$ incidence matrix of a BIB design with parameters $v_L, b_L, r_L, k_L, \lambda_L$ such that $v_2 = v_4$, $v_3 = v_5$ and $v_1 = v_2 + v_3$. Now we form the matrix N as

$$N = \left[\mathbf{1}'_{p_1} \otimes N_1 : \begin{matrix} \mathbf{1}'_{p_2} \otimes N_2 & J_{v_2 \times p_3 b_3} & \mathbf{1}'_{p_4} \otimes N_4 & O_{v_2 \times p_5 b_5} & I_{v_2} & O_{v_2 \times v_3} \\ J_{v_3 \times p_2 b_2} & \mathbf{1}'_{p_3} \otimes N_3 & O_{v_3 \times p_4 b_4} & \mathbf{1}'_{p_5} \otimes N_5 & O_{v_3 \times v_2} & I_{v_3} \end{matrix}\right] \quad (2.1)$$

Theorem 1: Block design with the incidence matrix N of the form (2.1) is the BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + p_2 b_2 + p_3 b_3 + p_4 b_4 + p_5 b_5 + v_2 + v_3$, $\mathbf{r}' = \{(p_1 r_1 + p_2 r_2 + p_3 b_3 + p_4 r_4 + 1)\mathbf{1}'_{v_2}, (p_1 r_1 + p_2 b_2 + p_3 r_3 + p_5 r_5 + 1)\mathbf{1}'_{v_3}\}$,$\mathbf{k}' = \{k_1 \mathbf{1}'_{p_1 b_1}, (k_2 + v_3)\mathbf{1}'_{p_2 b_2}, (k_3 + v_2)\mathbf{1}'_{p_3 b_3}, k_4 \mathbf{1}'_{p_4 b_4}, k_5 \mathbf{1}'_{p_5 b_5}, 1\mathbf{1}'_{v_2}, 1\mathbf{1}'_{v_3}\}$.

Proof: For the block design with incidence matrix N given in (2.1) we have C matrix as in (1.1).

$$C = \begin{bmatrix} (a_1 + s_1)I_{v_2} - s_1 J_{v_2} & -s_0 J_{v_2 \times v_3} \\ -s_0 J_{v_3 \times v_2} & (a_2 + s_2)I_{v_3} - s_2 J_{v_2} \end{bmatrix}$$

The diagonal and off-diagonal elements of $C(= c_{ij})$ matrix are respectively given by

$$c_{ij} = a_1 = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 r_2 (k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3 b_3 (k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{p_4 r_4 (k_4 - 1)}{k_4}$$

$$c_{ij} = a_2 = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 b_2 (k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3 r_3 (k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{p_5 r_5 (k_5 - 1)}{k_5}$$

and

$$c_{ij} = s_1 = \frac{p_1 \lambda_1}{k_1} + \frac{p_2 \lambda_2}{(k_2 + v_3)} + \frac{p_3 b_3}{(k_3 + v_2)} + \frac{p_4 \lambda_4}{k_4}$$

$$c_{ij} = s_0 = \frac{p_1 \lambda_1}{k_1} + \frac{p_2 r_2}{(k_2 + v_3)} + \frac{p_3 r_3}{(k_3 + v_2)}$$

$$c_{ij} = s_2 = \frac{p_1 \lambda_1}{k_1} + \frac{p_2 b_2}{(k_2 + v_3)} + \frac{p_3 \lambda_3}{(k_3 + v_2)} + \frac{p_5 \lambda_5}{k_5}$$

Then by Jaggi, Parsad and Gupta (1999),

$$\frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 r_2 (k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3 b_3 (k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{p_4 r_4 (k_4 - 1)}{k_4}$$

$$- (v_2 - 1)\left\{\frac{p_1 \lambda_1}{k_1} + \frac{p_2 \lambda_2}{(k_2 + v_3)} + \frac{p_3 b_3}{(k_3 + v_2)} + \frac{p_4 \lambda_4}{k_4}\right\} - v_3 \left\{\frac{p_1 \lambda_1}{k_1} + \frac{p_2 r_2}{(k_2 + v_3)} + \frac{p_3 r_3}{(k_3 + v_2)}\right\}$$

$$= 0$$

and

$$\frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 b_2 (k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3 r_3 (k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{p_5 r_5 (k_5 - 1)}{k_5}$$

$$- (v_3 - 1)\left\{\frac{p_1 \lambda_1}{k_1} + \frac{p_2 b_2}{(k_2 + v_3)} + \frac{p_3 \lambda_3}{(k_3 + v_2)} + \frac{p_5 \lambda_5}{k_5}\right\} - v_2 \left\{\frac{p_1 \lambda_1}{k_1} + \frac{p_2 r_2}{(k_2 + v_3)} + \frac{p_3 r_3}{(k_3 + v_2)}\right\}$$

$$= 0.$$

Hence the proof.

Example 2.1: Consider five BIB designs with parameters (9,12,4,3,1), (5,10,6,3,3), (4,6,3,2,1), (5,10,4,2,1) and (4,4,3,3,2) respectively. Then taking $p_1 = p_2 = p_3 = p_4 = 1$ and $p_5 = 2$, the design D with incidence matrix N as in (2.1) is a non-proper non-equireplicate BBPB design with parameters $v_1^* = 5$, $v_2^* = 4$, $b = 55$, $\boldsymbol{r}' = \{21\mathbf{1}_5', 24\mathbf{1}_4'\}$, $\boldsymbol{k}' = \{3\mathbf{1}_{12}', 7\mathbf{1}_{10}', 7\mathbf{1}_6', 2\mathbf{1}_{10}', 3\mathbf{1}_8', 1\mathbf{1}_5', 1\mathbf{1}_4'\}$.

Corollary 2.1: In Theorem 2.1, if we remove last v_2 and v_3 blocks, then we get a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1b_1 + p_2b_2 + p_3b_3 + p_4b_4 + p_5b_5$, $\boldsymbol{r}' = \{(p_1r_1 + p_2r_2 + p_3b_3 + p_4r_4)\mathbf{1}_{v_2}', (p_1r_1 + p_2b_2 + p_3r_3 + p_5r_5)\mathbf{1}_{v_3}'\}$,$\boldsymbol{k}' = \{k_1\mathbf{1}_{p_1b_1}', (k_2 + v_3)\mathbf{1}_{p_2b_2}', (k_3 + v_2)\mathbf{1}_{p_3b_3}', k_4\mathbf{1}_{p_4b_4}', k_5\mathbf{1}_{p_5b_5}'\}$.

Example 2.2: In Example 2.1, if we remove last v_2 and v_3 blocks, then we get a non-proper non-equireplicate BBPB design D with $p_1 = p_2 = p_3 = p_4 = 1$ and $p_5 = 2$. The parameters of the design are $v_1^* = 5$, $v_2^* = 4$, $b = 46$, $\boldsymbol{r}' = \{20\mathbf{1}_5', 23\mathbf{1}_4'\}$, $\boldsymbol{k}' = \{3\mathbf{1}_{12}', 7\mathbf{1}_{10}', 7\mathbf{1}_6', 2\mathbf{1}_{10}', 3\mathbf{1}_8'\}$.

Corollary 2.2: In Theorem 2.1, if we remove last p_4b_4, p_5b_5, v_2 and v_3 blocks, then again we get a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1b_1 + p_2b_2 + p_3b_3$, $\boldsymbol{r}' = \{(p_1r_1 + p_2r_2 + p_3b_3)\mathbf{1}_{v_2}', (p_1r_1 + p_2b_2 + p_3r_3)\mathbf{1}_{v_3}'\}$,$\boldsymbol{k}' = \{k_1\mathbf{1}_{p_1b_1}', (k_2 + v_3)\mathbf{1}_{p_2b_2}', (k_3 + v_2)\mathbf{1}_{p_3b_3}'\}$.

Example 2.3: Consider three BIB designs with parameters (9,12,4,3,1), (5,10,4,2,1) and (4,4,3,3,2) respectively. Then using Corollary 2.2 and taking $p_1 = 1$, $p_2 = 3$ and $p_3 = 2$, we get a non-proper non-equireplicate BBPB design D with parameters $v_1^* = 5$, $v_2^* = 4$, $b = 50$, $\boldsymbol{r}' = \{24\mathbf{1}_5', 40\mathbf{1}_4'\}$, $\boldsymbol{k}' = \{3\mathbf{1}_{12}', 6\mathbf{1}_{20}', 8\mathbf{1}_{18}'\}$.

Remark 2.1: In Corollary 2.2 if $p_2 = k_2 = k_3 = k$ and $p_3 = k_1$, then we get a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1b_1 + kb_2 + k_1b_3$, $\boldsymbol{r'} = \{(p_1r_1 + kr_2 + k_1b_3)\mathbf{1}_{v_2}', (p_1r_1 + kb_2 + k_1r_3)\mathbf{1}_{v_3}'\}$, $\boldsymbol{k'} = \{k_1\mathbf{1}_{p_1b_1}', (k + v_3)\mathbf{1}_{kb_2}', (k + v_2)\mathbf{1}_{k_1b_3}'\}$.

Example 2.4: Consider three BIB designs with parameters (9,12,4,3,1), (5,10,4,2,1) and (4,6,3,2,1) respectively. Then taking $p_1 = 1$, $p_2 = 2$ and $p_3 = 3$, we get a non-proper non-equireplicate BBPB design D with parameters $v_1^* = 5$, $v_2^* = 4$, $b = 50$, $\boldsymbol{r}' = \{30\mathbf{1}_5', 33\mathbf{1}_4'\}$, $\boldsymbol{k}' = \{3\mathbf{1}_{12}', 6\mathbf{1}_{20}', 7\mathbf{1}_{18}'\}$.

Remark 2.2: A special case of Corollary 2.2 arise when $p_1 = p_2 = p_3 = 1$. Then the resulting design is again a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = b_1 + b_2 + b_3$, $\boldsymbol{r}' = \{(r_1 + r_2 + b_3)\mathbf{1}_{v_2}', (r_1 + b_2 + r_3)\mathbf{1}_{v_3}'\}$, $\boldsymbol{k}' = \{k_1\mathbf{1}_{b_1}', (k_2 + v_3)\mathbf{1}_{b_2}', (k_3 + v_2)\mathbf{1}_{b_3}'\}$.

Example 2.5: Consider three BIB designs with parameters (8,14,7,4,3), (5,10,4,2,1) and (3,3,2,2,1) respectively. Then we get a non-proper non-equireplicate BBPB design D with parameters $v_1^* = 5$, $v_2^* = 3$, $b = 27$, $\boldsymbol{r}' = \{14\mathbf{1}_5', 19\mathbf{1}_3'\}$, $\boldsymbol{k}' = \{4\mathbf{1}_{14}', 5\mathbf{1}_{10}', 7\mathbf{1}_3'\}$.

Remark 2.3: Following theorems can be proved on the similar lines of Theorem 2.1. So we avoided proofs of the Theorems.

Let N_L (L = 1,2,3,4,5) be the $v_L \times b_L$ incidence matrix of a BIB design with parameters $v_L, b_L, r_L, k_L, \lambda_L$ such that $v_2 = v_4$, $v_3 = v_5$ and $v_1 = v_2 + v_3$. Now we form the matrix N as

$$N = \left[\mathbf{1}_{p_1}' \otimes N_1 : \begin{matrix} \mathbf{1}_{p_2}' \otimes N_2 & J_{v_2 \times p_3b_3} & \mathbf{1}_{p_4}' \otimes N_4 & J_{v_2 \times p_5b_5} & I_{v_2} & O_{v_2 \times v_3} \\ J_{v_3 \times p_2b_2} & \mathbf{1}_{p_3}' \otimes N_3 & J_{v_3 \times p_4b_4} & \mathbf{1}_{p_5}' \otimes N_5 & O_{v_3 \times v_2} & I_{v_3} \end{matrix}\right] \quad (2.2)$$

Theorem 2.2: Block design with the incidence matrix N of the form (2.2) is the BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1b_1 + p_2b_2 + p_3b_3 + p_4b_4 + p_5b_5 + v_2 + v_3$, $\boldsymbol{r}' = \{(p_1r_1 + p_2r_2 + p_3b_3 + p_4r_4 + p_5b_5 + 1)\mathbf{1}_{v_2}', (p_1r_1 + p_2b_2 + p_3r_3 + p_4b_4 + p_5r_5 + 1)\mathbf{1}_{v_3}'\}$, $\boldsymbol{k}' = \{k_1\mathbf{1}_{p_1b_1}', (k_2 + v_3)\mathbf{1}_{p_2b_2}', (k_3 + v_2)\mathbf{1}_{p_3b_3}', (k_4 + v_3)\mathbf{1}_{p_4b_4}', (k_5 + v_2)\mathbf{1}_{p_5b_5}', 1\mathbf{1}_{v_2}', 1\mathbf{1}_{v_3}'\}$. The diagonal and off-diagonal elements of its $C(= c_{ij})$ matrix are respectively given by

$$c_{ij} = \frac{p_1r_1(k_1 - 1)}{k_1} + \frac{p_2r_2(k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3b_3(k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{p_4r_4(k_4 + v_3 - 1)}{(k_4 + v_3)} + \frac{p_5b_5(k_5 + v_2 - 1)}{(k_5 + v_2)},$$

$$c_{ij} = \frac{p_1r_1(k_1 - 1)}{k_1} + \frac{p_2b_2(k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3r_3(k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{p_4b_4(k_4 + v_3 - 1)}{(k_4 + v_3)} + \frac{p_5r_5(k_5 + v_2 - 1)}{(k_5 + v_2)}$$

and

$$c_{ij} = \frac{p_1\lambda_1}{k_1} + \frac{p_2\lambda_2}{(k_2+v_3)} + \frac{p_3b_3}{(k_3+v_2)} + \frac{p_4\lambda_4}{(k_4+v_3)} + \frac{p_5b_5}{(k_5+v_2)},$$

$$c_{ij} = \frac{p_1\lambda_1}{k_1} + \frac{p_2r_2}{(k_2+v_3)} + \frac{p_3r_3}{(k_3+v_2)} + \frac{p_4r_4}{(k_4+v_3)} + \frac{p_5r_5}{(k_5+v_2)},$$

$$c_{ij} = \frac{p_1\lambda_1}{k_1} + \frac{p_2b_2}{(k_2+v_3)} + \frac{p_3\lambda_3}{(k_3+v_2)} + \frac{p_4b_4}{(k_4+v_3)} + \frac{p_5\lambda_5}{(k_5+v_2)}.$$

Example 2.6: Consider five BIB designs with parameters (11,11,5,5,2), (7,7,4,4,2), (4,6,3,2,1), (7,7,3,3,1) and (4,4,3,3,2) respectively. Then taking $p_1 = p_2 = p_3 = p_4 = p_5 = 1$, the design D with incidence matrix N as in (2.2) is a non-proper non-equireplicate BBPB design with parameters $v_1^* = 7$, $v_2^* = 4$, $b = 46$, $\boldsymbol{r}' = \{23\mathbf{1}_7', 26\mathbf{1}_4'\}$, $\boldsymbol{k}' = \{5\mathbf{1}_{11}', 8\mathbf{1}_7', 9\mathbf{1}_6', 7\mathbf{1}_7', 10\mathbf{1}_4', 1\mathbf{1}_7', 1\mathbf{1}_4'\}$.

Corollary 2.3: In Theorem 2.2, if we remove last v_2 and v_3 blocks, then we get a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1b_1 + p_2b_2 + p_3b_3 + p_4b_4 + p_5b_5$, $\boldsymbol{r}' = \{(p_1r_1 + p_2r_2 + p_3b_3 + p_4r_4 + p_5b_5)\mathbf{1}_{v_2}', (p_1r_1 + p_2b_2 + p_3r_3 + p_4b_4 + p_5r_5)\mathbf{1}_{v_3}'\}$, $\boldsymbol{k}' = \{k_1\mathbf{1}_{p_1b_1}', (k_2+v_3)\mathbf{1}_{p_2b_2}', (k_3+v_2)\mathbf{1}_{p_3b_3}', (k_4+v_3)\mathbf{1}_{p_4b_4}', (k_5+v_2)\mathbf{1}_{p_5b_5}'\}$.

Example 2.7: In Example 2.6, if we remove last v_2 and v_3 blocks, then we get a non-proper non-equireplicate BBPB design D with $p_1 = p_2 = p_3 = p_4 = p_5 = 1$. The parameters of the design are $v_1^* = 7$, $v_2^* = 4$, $b = 35$, $\boldsymbol{r}' = \{22\mathbf{1}_7', 25\mathbf{1}_4'\}$, $\boldsymbol{k}' = \{5\mathbf{1}_{11}', 8\mathbf{1}_7', 9\mathbf{1}_6', 7\mathbf{1}_7', 10\mathbf{1}_4'\}$.

Let N_L (L = 1,2,3,4,5) be the $v_L \times b_L$ incidence matrix of a BIB design with parameters v_L, b_L, r_L, k_L, λ_L such that $v_2 = v_4$, $v_3 = v_5$ and $v_1 = v_2 + v_3$. Now we form the matrix N as

$$N = \left[\mathbf{1}_{p_1}' \otimes N_1 : \begin{matrix} \mathbf{1}_{p_2}' \otimes N_2 & J_{v_2 \times p_3b_3} & \mathbf{1}_{p_4}' \otimes N_4 & I_{v_2} \otimes \mathbf{1}_{b_5}' & I_{v_2} & O_{v_2 \times v_3} \\ J_{v_3 \times p_2b_2} & \mathbf{1}_{p_3}' \otimes N_3 & J_{v_3 \times p_4b_4} & \mathbf{1}_{v_2}' \otimes N_5 & O_{v_3 \times v_2} & I_{v_3} \end{matrix}\right] \quad (2.3)$$

Theorem 2.3: Block design with the incidence matrix N of the form (2.3) is the BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1b_1 + p_2b_2 + p_3b_3 + p_4b_4 + v_2b_5 + v_2 + v_3$, $\boldsymbol{r}' = \{(p_1r_1 + p_2r_2 + p_3b_3 + p_4r_4 + b_5 + 1)\mathbf{1}_{v_2}', (p_1r_1 + p_2b_2 + p_3r_3 + p_4b_4 + v_2r_5 + 1)\mathbf{1}_{v_3}'\}$, $\boldsymbol{k}' = \{k_1\mathbf{1}_{p_1b_1}', (k_2+v_3)\mathbf{1}_{p_2b_2}', (k_3+v_2)\mathbf{1}_{p_3b_3}', (k_4+v_3)\mathbf{1}_{p_4b_4}', (k_5+1)\mathbf{1}_{v_2b_5}', 1\mathbf{1}_{v_2}', 1\mathbf{1}_{v_3}'\}$. The diagonal and off-diagonal elements of its $C(= c_{ij})$ matrix are respectively given by

$$c_{ij} = \frac{p_1r_1(k_1-1)}{k_1} + \frac{p_2r_2(k_2+v_3-1)}{(k_2+v_3)} + \frac{p_3b_3(k_3+v_2-1)}{(k_3+v_2)} + \frac{p_4r_4(k_4+v_3-1)}{(k_4+v_3)} + \frac{b_5k_5}{(k_5+1)},$$

$$c_{ij} = \frac{p_1r_1(k_1-1)}{k_1} + \frac{p_2b_2(k_2+v_3-1)}{(k_2+v_3)} + \frac{p_3r_3(k_3+v_2-1)}{(k_3+v_2)} + \frac{p_4b_4(k_4+v_3-1)}{(k_4+v_3)} + \frac{v_2r_5k_5}{(k_5+1)}$$

and

$$c_{ij} = \frac{p_1\lambda_1}{k_1} + \frac{p_2\lambda_2}{(k_2+v_3)} + \frac{p_3b_3}{(k_3+v_2)} + \frac{p_4\lambda_4}{(k_4+v_3)},$$

$$c_{ij} = \frac{p_1\lambda_1}{k_1} + \frac{p_2r_2}{(k_2+v_3)} + \frac{p_3r_3}{(k_3+v_2)} + \frac{p_4r_4}{(k_4+v_3)} + \frac{r_5}{(k_5+1)},$$

$$c_{ij} = \frac{p_1\lambda_1}{k_1} + \frac{p_2b_2}{(k_2+v_3)} + \frac{p_3\lambda_3}{(k_3+v_2)} + \frac{p_4b_4}{(k_4+v_3)} + \frac{v_2\lambda_5}{(k_5+1)}.$$

Example 2.8: Consider five BIB designs with parameters (11,11,5,5,2), (6,15,5,2,1), (5,10,6,3,3), (6,6,5,5,4) and (5,5,4,4,3) respectively. Then taking $p_1 = p_2 = p_3 = p_4 = 1$, the design D with incidence matrix N as in (2.3) is a non-proper non-equireplicate BBPB design with parameters $v_1^* = 6$, $v_2^* = 5$, $b = 83$, $\boldsymbol{r}' = \{31\mathbf{1}_6', 57\mathbf{1}_5'\}$, $\boldsymbol{k}' = \{5\mathbf{1}_{11}', 7\mathbf{1}_{15}', 9\mathbf{1}_{10}', 10\mathbf{1}_6', 5\mathbf{1}_{30}', 1\mathbf{1}_6', 1\mathbf{1}_5'\}$.

Corollary 2.4: In Theorem 2.3, if we remove last v_2 and v_3 blocks, then we get a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1b_1 + p_2b_2 + p_3b_3 + p_4b_4 + v_2b_5$, $\boldsymbol{r}' = \{(p_1r_1 + p_2r_2 + p_3b_3 + p_4r_4 + b_5)\mathbf{1}_{v_2}', (p_1r_1 + p_2b_2 + p_3r_3 + p_4b_4 + v_2r_5)\mathbf{1}_{v_3}'\}$, $\boldsymbol{k}' = \{k_1\mathbf{1}_{p_1b_1}', (k_2+v_3)\mathbf{1}_{p_2b_2}', (k_3+v_2)\mathbf{1}_{p_3b_3}', (k_4+v_3)\mathbf{1}_{p_4b_4}', (k_5+1)\mathbf{1}_{v_2b_5}'\}$.

Example 2.9: In Example 2.8, if we remove last v_2 and v_3 blocks, then we get a non-proper non-equireplicate BBPB design D with $p_1 = p_2 = p_3 = p_4 = 1$. The parameters of the design are $v_1^* = 6$, $v_2^* = 5$, $b = 72$, $\boldsymbol{r}' = \{30\mathbf{1}_6', 56\mathbf{1}_5'\}$, $\boldsymbol{k}' = \{5\mathbf{1}_{11}', 7\mathbf{1}_{15}', 9\mathbf{1}_{10}', 10\mathbf{1}_6', 5\mathbf{1}_{30}'\}$.

Let N_L (L = 1,2,3,4,5) be the $v_L \times b_L$ incidence matrix of a BIB design with parameters $v_L, b_L, r_L, k_L, \lambda_L$ such that $v_2 = v_4$, $v_3 = v_5$ and $v_1 = v_2 + v_3$. Now we form the matrix N as

$$N = \left[\mathbf{1}_{p_1}' \otimes N_1 : \begin{matrix} \mathbf{1}_{p_2}' \otimes N_2 & J_{v_2 \times p_3 b_3} & \mathbf{1}_{p_4}' \otimes N_4 & J_{v_2 \times p_5 b_5} & I_{v_2} & O_{v_2 \times v_3} \\ J_{v_3 \times p_2 b_2} & \mathbf{1}_{p_3}' \otimes N_3 & O_{v_3 \times p_4 b_4} & \mathbf{1}_{p_5}' \otimes N_5 & O_{v_3 \times v_2} & I_{v_3} \end{matrix}\right] \quad (2.4)$$

Theorem 2.4: Block design with the incidence matrix N of the form (2.4) is the BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + p_2 b_2 + p_3 b_3 + p_4 b_4 + p_5 b_5 + v_2 + v_3$, $\boldsymbol{r}' = \{(p_1 r_1 + p_2 r_2 + p_3 b_3 + p_4 r_4 + p_5 b_5 + 1)\mathbf{1}_{v_2}', (p_1 r_1 + p_2 b_2 + p_3 r_3 + p_5 r_5 + 1)\mathbf{1}_{v_3}'\}$, $\boldsymbol{k}' = \{k_1 \mathbf{1}_{p_1 b_1}', (k_2 + v_3)\mathbf{1}_{p_2 b_2}', (k_3 + v_2)\mathbf{1}_{p_3 b_3}', k_4 \mathbf{1}_{p_4 b_4}', (k_5 + v_2)\mathbf{1}_{p_5 b_5}', 1\mathbf{1}_{v_2}', 1\mathbf{1}_{v_3}'\}$. The diagonal and off-diagonal elements of its $C(= c_{ij})$ matrix are respectively given by

$$c_{ij} = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 r_2 (k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3 b_3 (k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{p_4 r_4 (k_4 - 1)}{k_4} + \frac{p_5 b_5 (k_5 + v_2 - 1)}{(k_5 + v_2)},$$

$$c_{ij} = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 b_2 (k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3 r_3 (k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{p_5 r_5 (k_5 + v_2 - 1)}{(k_5 + v_2)}$$

and

$$c_{ij} = \frac{p_1 \lambda_1}{k_1} + \frac{p_2 \lambda_2}{(k_2 + v_3)} + \frac{p_3 b_3}{(k_3 + v_2)} + \frac{p_4 \lambda_4}{k_4} + \frac{p_5 b_5}{(k_5 + v_2)},$$

$$c_{ij} = \frac{p_1 \lambda_1}{k_1} + \frac{p_2 r_2}{(k_2 + v_3)} + \frac{p_3 r_3}{(k_3 + v_2)} + \frac{p_5 r_5}{(k_5 + v_2)},$$

$$c_{ij} = \frac{p_1 \lambda_1}{k_1} + \frac{p_2 b_2}{(k_2 + v_3)} + \frac{p_3 \lambda_3}{(k_3 + v_2)} + \frac{p_5 \lambda_5}{(k_5 + v_2)}.$$

Example 2.10: Consider five BIB designs with parameters (9,12,4,3,1), (5,5,4,4,3), (4,4,3,3,2), (5,10,6,3,3) and (4,6,3,2,1) respectively. Then taking $p_1 = p_4 = 1$ and $p_2 = p_3 = p_5 = 2$, the design D with incidence matrix N as in (2.4) is a non-proper non-equireplicate BBPB design with parameters $v_1^* = 5$, $v_2^* = 4$, $b = 61$, $\boldsymbol{r}' = \{39\mathbf{1}_5', 27\mathbf{1}_4'\}$, $\boldsymbol{k}' = \{3\mathbf{1}_{12}', 8\mathbf{1}_{10}', 8\mathbf{1}_8', 3\mathbf{1}_{10}', 7\mathbf{1}_{12}', 1\mathbf{1}_5', 1\mathbf{1}_4'\}$.

Corollary 2.5: In Theorem 2.4, if we remove last v_2 and v_3 blocks, then we get a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + p_2 b_2 + p_3 b_3 + p_4 b_4 + p_5 b_5$, $\boldsymbol{r}' = \{(p_1 r_1 + p_2 r_2 + p_3 b_3 + p_4 r_4 + p_5 b_5)\mathbf{1}_{v_2}', (p_1 r_1 + p_2 b_2 + p_3 r_3 + p_5 r_5)\mathbf{1}_{v_3}'\}$, $\boldsymbol{k}' = \{k_1 \mathbf{1}_{p_1 b_1}', (k_2 + v_3)\mathbf{1}_{p_2 b_2}', (k_3 + v_2)\mathbf{1}_{p_3 b_3}', k_4 \mathbf{1}_{p_4 b_4}', (k_5 + v_2)\mathbf{1}_{p_5 b_5}'\}$.

Example 2.11: In Example 2.10, if we remove last v_2 and v_3 blocks, then we get a non-proper non-equireplicate BBPB design D with $p_1 = p_4 = 1$ and $p_2 = p_3 = p_5 = 2$. The parameters of the design are $v_1^* = 5$, $v_2^* = 4$, $b = 61$, $\boldsymbol{r}' = \{38\mathbf{1}_5', 26\mathbf{1}_4'\}$, $\boldsymbol{k}' = \{3\mathbf{1}_{12}', 8\mathbf{1}_{10}', 8\mathbf{1}_8', 3\mathbf{1}_{10}', 7\mathbf{1}_{12}'\}$.

Let N_L (L = 1,2,3,4,5) be the $v_L \times b_L$ incidence matrix of a BIB design with parameters $v_L, b_L, r_L, k_L, \lambda_L$ such that $v_2 = v_4$, $v_3 = v_5$ and $v_1 = v_2 + v_3$. Now we form the matrix N as

$$N = \left[\mathbf{1}_{p_1}' \otimes N_1 : \begin{matrix} \mathbf{1}_{p_2}' \otimes N_2 & J_{v_2 \times p_3 b_3} & \mathbf{1}_{p_4}' \otimes N_4 & I_{v_2} \otimes \mathbf{1}_{b_5}' & I_{v_2} & O_{v_2 \times v_3} \\ J_{v_3 \times p_2 b_2} & \mathbf{1}_{p_3}' \otimes N_3 & O_{v_3 \times p_4 b_4} & \mathbf{1}_{v_2}' \otimes N_5 & O_{v_3 \times v_2} & I_{v_3} \end{matrix}\right] \quad (2.5)$$

Theorem 2.5: Block design with the incidence matrix N of the form (2.5) is the BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + p_2 b_2 + p_3 b_3 + p_4 b_4 + v_2 b_5 + v_2 + v_3$, $\boldsymbol{r}' = \{(p_1 r_1 + p_2 r_2 + p_3 b_3 + p_4 r_4 + b_5 + 1)\mathbf{1}_{v_2}', (p_1 r_1 + p_2 b_2 + p_3 r_3 + v_2 r_5 + 1)\mathbf{1}_{v_3}'\}$, $\boldsymbol{k}' = \{k_1 \mathbf{1}_{p_1 b_1}', (k_2 + v_3)\mathbf{1}_{p_2 b_2}', (k_3 + v_2)\mathbf{1}_{p_3 b_3}', k_4 \mathbf{1}_{p_4 b_4}', (k_5 + 1)\mathbf{1}_{v_2 b_5}', 1\mathbf{1}_{v_2}', 1\mathbf{1}_{v_3}'\}$. The diagonal and off-diagonal elements of its $C(= c_{ij})$ matrix are respectively given by

$$c_{ij} = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 r_2 (k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3 b_3 (k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{p_4 r_4 (k_4 - 1)}{k_4} + \frac{b_5 k_5}{(k_5 + 1)},$$

$$c_{ij} = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 b_2 (k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3 r_3 (k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{v_2 r_5 k_5}{(k_5 + 1)}$$

and

$$c_{ij} = \frac{p_1 \lambda_1}{k_1} + \frac{p_2 \lambda_2}{(k_2 + v_3)} + \frac{p_3 b_3}{(k_3 + v_2)} + \frac{p_4 \lambda_4}{k_4},$$

$$c_{ij} = \frac{p_1 \lambda_1}{k_1} + \frac{p_2 r_2}{(k_2 + v_3)} + \frac{p_3 r_3}{(k_3 + v_2)} + \frac{r_5}{(k_5 + 1)},$$

$$c_{ij} = \frac{p_1 \lambda_1}{k_1} + \frac{p_2 b_2}{(k_2 + v_3)} + \frac{p_3 \lambda_3}{(k_3 + v_2)} + \frac{v_2 \lambda_5}{(k_5 + 1)}.$$

Example 2.12: Consider five BIB designs with parameters (11,11,5,5,2), (7,7,4,4,2), (4,4,3,3,2), (7,7,3,3,1) and (4,6,3,2,1) respectively. Then taking $p_1 = p_2 = p_4 = 1$ and $p_3 = 4$, the design D with incidence matrix N as in (2.5) is a non-proper non-equireplicate BBPB design with parameters $v_1^* = 7$, $v_2^* = 4$, $b = 94$, $\boldsymbol{r}' = \{35\mathbf{1}_7', 45\mathbf{1}_4'\}$, $\boldsymbol{k}' = \{5\mathbf{1}_{11}', 8\mathbf{1}_7', 10\mathbf{1}_{16}', 3\mathbf{1}_7', 3\mathbf{1}_{42}', 1\mathbf{1}_7', 1\mathbf{1}_4'\}$.

Corollary 2.6: In Theorem 2.5, if we remove last v_2 and v_3 blocks, then we get a BBPB design D with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + p_2 b_2 + p_3 b_3 + p_4 b_4 + v_2 b_5$, $\boldsymbol{r}' = \{(p_1 r_1 + p_2 r_2 + p_3 b_3 + p_4 r_4 + b_5)\mathbf{1}_{v_2}', (p_1 r_1 + p_2 b_2 + p_3 r_3 + v_2 r_5)\mathbf{1}_{v_3}'\}$, $\boldsymbol{k}' = \{k_1 \mathbf{1}_{p_1 b_1}', (k_2 + v_3)\mathbf{1}_{p_2 b_2}', (k_3 + v_2)\mathbf{1}_{p_3 b_3}', k_4 \mathbf{1}_{p_4 b_4}', (k_5 + 1)\mathbf{1}_{v_2 b_5}'\}$.

Example 2.13: In Example 2.12, if we remove last v_2 and v_3 blocks, then we get a non-proper non-equireplicate BBPB design D with $p_1 = p_2 = p_4 = 1$ and $p_3 = 4$. The parameters of the design are $v_1^* = 7$, $v_2^* = 4$, $b = 83$, $\boldsymbol{r}' = \{34\mathbf{1}_7', 44\mathbf{1}_4'\}$, $\boldsymbol{k}' = \{5\mathbf{1}_{11}', 8\mathbf{1}_7', 10\mathbf{1}_{16}', 3\mathbf{1}_7', 3\mathbf{1}_{42}'\}$.

3. Conclusion

Here for comparison of a set of test treatments to a set of control treatments balanced bipartite block designs with unequal block sizes are derived by the new methods of construction. Such designs can be applied in pharmaceutical, industrial and agricultural experiments. The methods are flexible enough to incorporate number of incidence matrices of BIB designs.

References

1. Angelis, L. and Moyssiadis, C., A-optimal incomplete block designs with unequal block sizes for comparing test treatments with a control, *J. Statist. Plan. Infer.*, 28, 353-368 (1991)
2. Angelis, L., Moyssiadis, C. and Kageyama, S., Methods of constructing A-efficient BTIUB designs, *Utilitas Math*., 44, 5-15 (1993)
3. Bechhofer, R.E. and Tamhane, A.C., Incomplete block designs for comparing treatments with a control : General theory, *Technometrics*, 23, 45-57 (1981)
4. Dey, A., Incomplete Block Designs, *Hindustan Book Agency*, New Delhi (2010)
5. Gupta, V.K. and Parsad, R., Block designs for comparing test treatments with control treatments: An overview, *Statist. Appln*., 3, 133-146 (2001)
6. Jacroux, M., On comparing test treatments with a control using block designs having unequal sized blocks, *Sankhya*, B, 54, 324-345 (1992)
7. Jacroux, M., On the determination and construction of A- and MV-optimal block designs for comparing a set of test treatments to set of standard treatments, *J. Statist. Plann. Inference*, 106, 191-204 (2002)
8. Jacroux, M., Some optimal orthogonal and nearly orthogonal block designs for comparing a set of test treatments to a set of standard treatments, *Sankhya Ser. B*, 62, 276-289 (2000)
9. Jaggi S., Parsad, R. and Gupta, V.K., Construction of non-proper balanced bipartite block designs, *Cal. Statist. Asso. Bull*., 49, 55-63 (1999)
10. Kageyama, S. and Sinha, K., Some constructions of balanced bipartite block designs, *Utilitas Math*., 33, 137-162 (1988)
11. Majumdar, D., Optimal designs for comparisons between two sets of treatments, *J. Statist. Plann. Inference*, 14, 359-372 (1986)
12. Parsad, R. and Gupta, V.K., Optimal block designs with unequal block sized for making test treatments control comparisons under a heteroscedastic model, *Sankhya*, B, 56, 449-461 (1994)
13. Parsad, R., Gupta, V.K. and Singh, V.P.N., Trace optimal block designs with unequal block sizes for comparing two disjoint sets of treatments, *Sankhya Ser. B*, 58, 414-421 (1996)
14. Sinha, K. and Kageyama, S., Further constructions of balanced bipartite block designs, *Utilitas Math*., 38, 155-160 (1990)

